COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 1

A VARIANT OF A THEOREM OF SIERPIŃSKI CONCERNING PARTITIONS OF CONTINUA

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By a *closed partition* of a topological space X we mean a family of non-empty, disjoint, closed sets whose union is X. A result of Sierpiński [3], reproduced in [1], § 42, III, 6, states that a compact, connected Hausdorff space has no countable, closed partition (where "countable" means "countably infinite"). In this note, we show how Baire's theorem can be applied to give a variant of this theorem in which the space is assumed to be locally connected but not compact.

THEOREM. *Let X be a connected, locally connected topological space in which each closed subset is non-meagre in itself. Then X has no countable closed partition.*

Proof. Suppose that

$$X = \bigcup_{n=1}^{\infty} A_n,$$

where the A_n are non-empty, disjoint and closed. Let B_n be the boundary of A_n; this is non-empty, since X is connected. Let

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Then B is closed, since its complement is the union of the interiors of the A_n. By hypothesis, there exists k such that B_k has a B-interior point x_0. There is a connected X-neighbourhood V of x_0 such that $V \cap B \subseteq B_k$, in other words, $V \cap B_n = \emptyset$ for all $n \neq k$. We show that

$$(1) \qquad V \cap A_n = \emptyset \quad \text{for all } n \neq k.$$

If $V \cap A_n$ is not empty, then it is a V-closed proper subset of V. Hence it is not also V-open (since V is connected), so it contains a point y such that every V-neighbourhood of y meets $V \setminus A_n$. But then $y \in B_n$ (the boundary of A_n), contradicting the fact that $V \cap B_n = \emptyset$.

Hence (1) is true. Since the A_n cover X, it follows that $V \subseteq A_k$. But this contradicts the fact that x_0 is a boundary point of A_k.

COROLLARY 1. *Let X be a locally connected space in which each closed subset is non-meagre in itself. Suppose that $\{A_n : n = 1, 2, \ldots\}$ is a closed partition of X. Then each component of X is contained in some A_n, and each A_n is open.*

Proof. Let C be a component. Then C is open-closed and locally connected. The sets $C \cap A_n$ cover C, so by the theorem, C is contained in some A_n. This shows that each A_n is a union of components, and therefore open.

The next corollary shows that connectedness can be replaced by local connectedness in Sierpiński's theorem:

COROLLARY 2. *A compact, locally connected, Hausdorff space has no countable, closed partition.*

Proof. By Baire's theorem, each closed subset is non-meagre in itself. If $\{A_n\}$ is a countable closed partion, then, by corollary 1, each A_n is open. This contradicts compactness.

Remarks. By Baire's theorem, two classes of spaces having the property that every closed subset is non-meagre in itself are (i) complete metric, and (ii) locally compact, Hausdorff spaces. In both cases, it is actually possible to deduce from Sierpiński's theorem that the space has no countable, closed partition — so the main applications of our theorem amount to new proofs of known results. In case (ii), the deduction from Sierpiński's theorem is quite simple, but in case (i) it uses the highly non-trivial fact that a complete, metric, connected, locally connected space is path-connected ([1], § 45, II, 1).

The example given by Mazurkiewicz in [2] shows that local connectedness cannot be dropped from the hypotheses of our theorem.

REFERENCES

[1] K. Kuratowski, *Topologie*, Warszawa 1958.

[2] S. Mazurkiewicz, *Sur les continus plans non bornés*, Fundamenta Mathematicae 5 (1924), p. 188-205.

[3] W. Sierpiński, *Un théoreme sur les continus*, Tohoku Mathematical Journal 13 (1918), p. 300-303.

Reçu par la Rédaction le 10. 5. 1971