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A CHARACTERIZATION OF THE DOMAIN OF ATTRACTION OF A NORMAL DISTRIBUTION IN A HILBERT SPACE

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This paper gives the necessary and sufficient conditions for a probability distribution in a Hilbert space to be attracted by a Gaussian measure. In paper [2] these conditions have been formulated in terms of certain operators with finite trace, whereas the characterization given here is connected with the distribution of the norm.

Let H be a separable real Hilbert space. Denote by $\mathfrak{M}$ the set of all probability measures in H, i.e. the set of normed regular measures defined on the σ-field $\mathscr{B}$ of all Borel subsets of H. $\mathfrak{M}$ is a complete space with the Lévy-Prokhorov metric (see [3], p. 188). Convergence in this metric space is equivalent to the weak convergence of distributions.

Let us recall that a sequence of distributions p_n is said to be *weakly convergent* to a distribution p if, for every continuous function f bounded in H, we have

$$\lim_{n\to\infty} \int_H f(h) p_n(dh) = \int_H f(h) p(dh).$$

The *convolution of distributions* p and q, which is defined by the formula

$$p*q(A) = \int_H p(A-h)q(dh) \quad \text{for every } A \in \mathscr{B},$$

is a continuous operation in $\mathfrak{M}$. The n-th convolution power of a distribution p will be denoted by p^{n*}.

A sequence of distributions $\{p_n\}$ is called *shift-convergent* if there exists a sequence $\{x_n\}$ of elements of H such that the sequence of distributions $\{p_n * \delta_{x_n}\}$ is convergent in $\mathfrak{M}$, where δ_x denotes the *measure condensed at a point* $x \in H$, i.e. $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$ for every $A \in \mathscr{B}$.

A linear operator in H is called an *S-operator* if it is non-negative, self-adjoint and has a finite trace (see [3], p. 193).

The *characteristic functional* $\hat{p}$ of a distribution p is defined by the formula

$$\hat{p}(f) = \int_H e^{i(x,f)} p(dx), \quad f \in H.$$

A distribution q is called *normal* if

$$\hat{q}(f) = \exp[i(x_0, f) - \tfrac{1}{2}(Df, f)],$$

where $x_0 \in H$ and D is an S-operator.

For every positive a and every distribution p in H, we put by definition

$$T_a p(A) = p\{x \in H: ax \in A\} \quad \text{for every } A \in \mathscr{B}.$$

The set of distribution $p \in \mathfrak{M}$ for which there exists a sequence of positive numbers $\{a_n\}$ such that the sequence of distributions $\{T_{a_n} p^{n*}\}$ is shift-convergent to a distribution q is called the *domain of attraction* of the distribution q.

Denote by Π the domain of attraction of a normal non-degenerate distribution in H.

Assign with a distribution p in H the distribution $\tilde{p}$ on the real line defined by the formula

$$\tilde{p}(B) = p\{x \in H: \|x\| \in B\} \tag{1}$$

for every Borel set B on the real line and the family of S-operators D_X defined by the bilinear form

(2)

$$(D_X g, h) = \int_{\|x\| \leqslant X} (x, g)(x, h) p(dx) - \int_{\|x\| \leqslant X} (x, g) p(dx) \cdot \int_{\|x\| \leqslant X} (x, h) p(dx),$$

and the numbers σ_X^2 defined by the formula

$$\sigma_X^2 = \int_{\|x\| \leqslant X} \|x\|^2 p(dx) - \Big[\int_{\|x\| \leqslant X} \|x\| p(dx)\Big]^2. \tag{3}$$

Let $k(n)$ be an arbitrary sequence of natural numbers such that $\lim_{n\to\infty} k(n) = +\infty$ and let $\{x_n\}$ be an arbitrary sequence of positive numbers such that $\lim_{n\to\infty} x_n = +\infty$.

LEMMA. *A sequence of distributions $\{T_{1/x_n} p^{k(n)*}\}$ is shift-convergent to a normal distribution in H if and only if for every $\varepsilon > 0$ the following conditions are satisfied:*

$$\lim_{n\to\infty} k(n) \int_{\|x\| \geqslant \varepsilon} T_{1/x_n} p(dx) = 0, \tag{4}$$

$$\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} k(n) \frac{1}{x_n^2} (D_{\varepsilon x_n} e_i, e_i) = 0, \tag{5}$$

where $\{e_i\}$ is a basis in H,

$$\lim_{n\to\infty} k(n) \frac{1}{x_n^2} (D_{\varepsilon x_n} f, f) = (Df, f) \quad \textit{for every } f \in H. \tag{6}$$

The quadratic form (Df, f) defines then the S-operator D which determines the limit normal distribution.

This lemma is a corollary to theorem 2.1 in [2].

THEOREM. *A non-degenerate distribution p belongs to Π if and only if the following conditions are satisfied:*

$$\text{(i)} \qquad \lim_{X\to+\infty} \frac{X^2 \int\limits_{\|x\|\geqslant X} p(dx)}{\int\limits_{\|x\|\leqslant X} \|x\|^2 p(dx)} = 0,$$

i.e. the distribution $\tilde{p}$ defined in (1) *is attracted by a non-degenerate normal distribution on the real line* (see theorem 1, § 34 in [1]);

$$\text{(ii)} \qquad \lim_{N\to\infty} \sup_{X>X_0} \sum_{i=N}^{\infty} \frac{(D_X e_i, e_i)}{\sigma_X^2} = 0,$$

where $\{e_i\}$ is a basis in H, and X_0 is a positive number;

$$\text{(iii)} \qquad \lim_{X\to\infty} \frac{(D_X f, f)}{\sigma_X^2} = (Sf, f) \quad \textit{for every } f \in H.$$

Then the quadratic form (Sf, f) determines, up to a constant factor, the S-operator characterizing the limit normal distribution.

Proof. Necessity. Let $\{a_n\}$ be a sequence of positive numbers for which the sequence of distributions $\{T_{a_n} p^{n*}\}$ is shift-convergent to a proper normal distribution in H determined by the S-operator D. Let $\{x_n\}$ be an arbitrary sequence of positive numbers such that $\lim\limits_{n\to\infty} x_n = +\infty$. Since

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1,$$

for every sufficiently great natural number n, a natural number $k(n)$ may be found such that (see lemma 3.1 in [2])

$$a_{k(n)+1} \leqslant \frac{1}{x_n} < a_{k(n)}$$

and thus such that

$$\text{(7)} \qquad \lim_{n\to\infty} x_n a_{k(n)} = 1.$$

It follows from (7) and from theorem 1.10 in [3] that the sequence of distributions $\{T_{1/x_n} p^{k(n)*}\}$ is shift-convergent to the normal distribution determined by the S-operator D. Thus conditions (4), (5) and (6) of the lemma are satisfied. Simultaneously, it follows from the assumption that the sequence of distributions $\{T_{a_n}\tilde{p}^{n*}\}$ is shift-convergent on the real line to a normal distribution with the variation

$$0 \neq \sum_{i=1}^{\infty}(De_i, e_i) = a^2 \neq +\infty$$

(see the proof of theorem 3.2 in [2]). Thus condition (i) is satisfied and, by (7), the sequence of distributions $\{T_{1/x_n}\tilde{p}^{k(n)*}\}$ on the real line is shift-convergent to a normal distribution with the variation a^2. By theorem 2, § 26 in [1], we obtain

$$\lim_{n\to\infty} k(n)\frac{1}{x_n^2}\sigma_{\varepsilon x_n}^2 = a^2 \neq 0 \quad \text{for every } \varepsilon > 0. \tag{8}$$

Condition (ii) follows from conditions (5) and (8) while (iii) follows from (6) and (8).

Sufficiency. It follows from (i) that the distribution $\tilde{p}$ is attracted by a non-degenerate normal distribution on the real line, and hence by theorem 2, § 26 in [1], for some sequence of positive numbers $\{a_n\}$ the following conditions analogous to (4) and (8) are satisfied for every $\varepsilon > 0$:

$$\lim_{n\to\infty} n \int_{\|x\|\geqslant\varepsilon} T_{a_n}p(dx) = 0, \tag{4'}$$

$$\lim_{n\to\infty} na_n^2\sigma_{\varepsilon/a_n}^2 = a^2 \neq 0. \tag{8'}$$

From assumptions (ii) and (iii) and from condition (8) we obtain for every $\varepsilon > 0$ the conditions analogous to (5) and (6):

$$\lim_{N\to\infty}\sup_n \sum_{i=N}^{\infty} na_n^2(D_{\varepsilon/a_n}e_i, e_i) = 0, \tag{5'}$$

where $\{e_i\}$ is a basis in H;

$$\lim_{n\to\infty} na_n(D_{\varepsilon/a_n}f, f) = a^2(Sf, f) \quad \text{for every } f \in H. \tag{6'}$$

By the lemma the sequence of distributions $\{T_{a_n}p^{n*}\}$ is shift-convergent to a normal distribution in H determined by the S-operator $D = a^2S$. It is easily seen that $(Df, f) \neq 0$, i.e. the limit normal distribution is non-degenerate.

REFERENCES

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