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ON TRANSFINITE INDUCTIVE COMPACTNESS DEGREE

BY

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1. The transfinite inductive compactness degree. We assume in this note that the spaces are metrizable and separable. We shall consider the extension by transfinite induction of the inductive compactness degree cmp defined by J. de Groot [4], [6; Research Problem D, p. 121]:

cmp $X = -1$ if X is compact, cmp $X \leqslant \alpha$, α being an ordinal number, if each point in X has arbitrarily small neighbourhoods V whose boundaries Fr V have cmp Fr $V < \alpha$, and we let cmp X be the first ordinal α with cmp $X \leqslant \alpha$, if such an α exists, or cmp $X = \infty$, in the opposite case.

Since our spaces X have always a countable base, cmp $X < \omega_1$, provided that cmp $X \neq \infty$.

Remark. If X is countable-dimensional, i.e. X is a union of countably many zero-dimensional sets, then cmp $X \neq \infty$ if and only if the small transfinite dimension ind X is defined, and then cmp $X \leqslant$ ind X, cf. [5; Ch. IV, § 6 (B)], [3]; this can be easily verified by induction, see [5; Ch. IV, § 6 (D)].

2. Main result. In this section we state the main result of this note; the proof will be given in Section 4, preceded by a discussion of some auxiliary facts, given in Section 3.

2.1. THEOREM. *For each countable ordinal number α there exists a separable, metrizable space C_α such that $\alpha \leqslant$ cmp $C_\alpha \neq \infty$.*

Each space C_α contains a compact subspace S_α with a locally compact complement $T_\alpha = C_\alpha \setminus S_\alpha$ such that S_α has a base of open-and-closed neighbourhoods with finite-dimensional complements in C_α.

The space C_α has only countably many components and the components of S_α are finite-dimensional closed cells, while the components of T_α are finite-dimensional open cells.

2.2. Remark. Let Cmp and Com be extensions by transfinite induction of the topological invariants considered respectively by de Groot

[4], [6; p. 121] and de Vries [13; 3.4.9], [8], and let Ind be the large transfinite inductive dimension [9; VI.3], [3]. Then, the structure of the spaces C_α desribed in Theorem 2.1 guarantees that Ind C_α is defined [3; 3.16] and

$$\alpha \leqslant \operatorname{cmp} C_\alpha \leqslant \operatorname{Cmp} C_\alpha \leqslant \operatorname{Com} C_\alpha \leqslant \operatorname{Ind} C_\alpha \neq \infty .$$

3. **Auxiliary facts.** In this section we collect some information which we shall use next in the proof of Theorem 2.1.

3.1. **The Effros Borel structure.** Given a space E we denote by $\boldsymbol{F}E$ the collection of all closed subsets of E endowed with the Effros Borel structure $\boldsymbol{B}$, i.e. $\boldsymbol{B}$ is the σ-algebra in $\boldsymbol{F}E$ generated by the sets $\{X \in \boldsymbol{F}E\colon X \cap U \neq \emptyset\}$, U being an open set in E. Recall [1; Ch. 3], that for any totally bounded metric inducing the topology in E, if dist is the corresponding Hausdorff distance in $\boldsymbol{F}E$ [7; § 21, VII], then $\boldsymbol{B}$ is the family of all Borel sets in the metric space $(\boldsymbol{F}E, \text{dist})$. The space E we shall consider later on will be σ-compact, and then $(\boldsymbol{F}E, \text{dist})$ is always an absolutely Borel space, which allows one to speak about analytic sets in $\boldsymbol{F}E$ without any ambiguity [1; Th. 3.2] and, moreover, in this case the intersection operation $\langle X, Y\rangle \to X \cap Y$ is a measurable map from the product space $(\boldsymbol{F}E \times \boldsymbol{F}E, \boldsymbol{B} \otimes \boldsymbol{B})$ to the space $(\boldsymbol{F}E, \boldsymbol{B})$ [1; Th. 3.10].

3.2. Lemma. *Let E be a space containing a compact set S with a locally compact complement $T = E \setminus S$ and let $f\colon E \to I$ be a continuous map onto the unit interval I. Then, for each countable ordinal α, the sets*

$$\boldsymbol{F}_\alpha = \{X \in \boldsymbol{F}E\colon \operatorname{cmp} X \leqslant \alpha\},$$
$$\boldsymbol{I}_\alpha = \{X \in \boldsymbol{F}I\colon \operatorname{cmp} f^{-1}(X) \leqslant \alpha\}$$

are analytic with respect to the Effros Borel structure.

Proof. A closed set L is a partition in E into two disjoint closed sets A and B if $E \setminus L$ is a disjoint union of two open sets V, W such that $A \subset V$ and $B \subset W$ ([2]). Since the intersection is a measurable operation (E being σ-compact), the set $L(A, B)$ of all partitions in E into the pair $\langle A, B\rangle$ of disjoint closed sets is analytic in $(\boldsymbol{F}E, \boldsymbol{B})$ (indeed, $L(A, B) = \operatorname{proj}\{\langle L, X, Y\rangle \in \boldsymbol{F}E \times \boldsymbol{F}E \times \boldsymbol{F}E\colon A \subset X,\ B \subset Y,\ L \subset X \cap Y,\ L \cap (A \cup B) = \emptyset,\ X \cup Y = E\}$).

We shall check that the sets $\boldsymbol{F}_\alpha$ are analytic by transfinite induction: $\boldsymbol{F}_{-1} = \{X \in \boldsymbol{F}E\colon X \text{ is compact}\} \in \boldsymbol{B}$, E being topologically complete; assume that $\boldsymbol{F}_\beta$ is analytic for $\beta < \alpha$ and let us consider $\boldsymbol{F}_\alpha$. Let us choose in each of the spaces S and T countable bases whose members have compact closures and let $\mathscr{K}$ be the collection of the closures of these sets; let us choose also a countable base in the space E and let $\mathscr{U}$ be the collection of all finite unions of its members. Given $K_i \in \mathscr{K}$ for $i \leqslant m$ and an open set $U \in \mathscr{U}$ such that

$\bigcup_{i=1}^{m} K_i \subset U$, we put

(1) $F(K_1, \ldots, K_m; U) = \{X \in FE:$

$$(\exists \langle L_1, \ldots, L_m\rangle \in \prod_{i=1}^{m} L(K_i, E \setminus U))(\forall i \leqslant m)\operatorname{cmp}(L_i \cap X) \leqslant \alpha\}.$$

Then, since the set $\{\langle X, L\rangle: L \cap X \in \bigcup_{\beta<\alpha} F_\beta\}$ is analytic by the inductive assumption, the set defined in (1) is a projection of an analytic set, and hence it is analytic. To complete the proof, it remains to observe that

$$F_\alpha = \bigcap_{K \in \mathscr{K}} \bigcap_{\substack{U \in \mathscr{U} \\ K \subset U}} \bigcup \{F(K_1, \ldots, K_m; U):$$

$$\langle K_1, \ldots, K_m\rangle \in K^m, m = 1, 2, \ldots, K \subset \bigcup_{i=1}^{m} K_i \subset U\}.$$

To see that each I_α is analytic, let us notice that, since E is σ-compact, the map $X \to f^{-1}(X)$ from FI to FE is measurable with respect to the Effros Borel structure in these spaces, and that I_α is the preimage under this map of the analytic set F_α.

3.3. The space C_∞. Let I be the unit interval $[0, 1]$, $I^\omega = I \times I \times \ldots$ be the Hilbert cube, let $I^n = \{(x_i) \in I^\omega: x_{n+1} = x_{n+2} = \ldots = 0\}$ be the n-dimensional cube, and let $\partial I^n = \{(x_i) \in I^n$: for some $i \leqslant n$, $x_i = 0$ or $x_i = 1\}$ be its combinatorial boundary. Let us define

(2) $$C_\infty = I^\omega \times \{0\} \cup \bigcup_{n=1}^{\infty} (I^n \setminus \partial I^n) \times \{1/n\}.$$

Elżbieta Pol [10; Example 6.1] proved that

(3) $$\operatorname{cmp} C_\infty = \chi.$$

4. Proof of Theorem 2.1. The reasoning in this section is a version of a reasoning from [11; § 2, Sec. 3], where some problems about transfinite inductive dimensions were considered. We adopt the notation introduced in Section 3.3; let $p_n: I^\omega \to I^n$ be the projection $p_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots)$ and let Q be the set of rational numbers from the interval I.

4.1. Let us arrange the rational numbers from I into a sequence $q_1, q_2, \ldots$, let us consider the product $I \times C_\infty$ (see 3.3 (2)), and let us attach to each compactum $\{q_n\} \times I^\omega \times \{0\}$ the cube I^n by the map p_n; to be more specific, we define an upper semi-continuous decomposition of the product $I \times C_\infty$ whose non-one-point members are the sets $\{q_n\} \times p_n^{-1}(x) \times \{0\}$, where $n = 1, 2, \ldots$ and $x \in I^n$ (notice that diam $p_n^{-1}(x) \leqslant 1/n$ in the standard metric in

I^ω). Let E be the resulting decomposition space and let $q\colon I\times C_\infty \to E$ be the quotient map. Let us observe that for each n, the restriction $q|I\times(I^n\setminus \partial I^n)\times\{1/n\}$ is a homeomorphism onto an open subspace E_n of E, the sets E_n are pairwise disjoint, and each neighbourhood of the compactum $E_0 = q(I\times I^\omega\times\{0\})$ contains all but finitely many sets E_n.

4.2. Let $f\colon E\to I$ be the continuous map induced by the projection pr: $I\times C_\infty \to I$, i.e. $\mathrm{pr} = f\circ q$. Then

(A) if $t\in I$ is irrational, then $f^{-1}(t)$ is homeomorphic to C_∞,

(B) if $X\subset Q$ is a compact subset of rational numbers, then the components of the compactum $f^{-1}(X)\cap E_0$ are the sets $f^{-1}(q_n)\cap E_0 \underset{\text{top}}{=} I^n$, for $q_n\in X$, and each set $f^{-1}(X)\cap E_n$ is homeomorphic to $X\times(I^n\setminus\partial I^n)$.

Therefore, for each compact set $X\subset Q$, the space $C = f^{-1}(X)$ has the structure described in Theorem 2.1 ($S = f^{-1}(X)\cap E_0$ being the compact subspace of C with the required properties).

4.3. We shall show that for each countable ordinal α there exists a compact subset X_α of rationals Q such that $\operatorname{cmp} f^{-1}(X_\alpha)\geqslant\alpha$. Then, the spaces $C_\alpha = f^{-1}(X_\alpha)$ will satisfy the assertion of Theorem 2.1, by 4.2 (B).

Let us adopt the notation of Lemma 3.2. Each compactum $X\in \boldsymbol{I}_\alpha$ is contained in Q, as in the opposite case, X would contain a closed set $f^{-1}(t)$ for an irrational $t\in I$, but then 4.2 (A) and 3.3 (3) would imply that $\operatorname{cmp} f^{-1}(X)\geqslant \operatorname{cmp} f^{-1}(t) = \infty$, contradicting the fact that $\operatorname{cmp} f^{-1}(X)\leqslant\alpha$. Therefore, $\boldsymbol{I}_\alpha$ is an analytic subset of the set $\boldsymbol{Q}$ of all compact subsets of Q, which is not analytic by the Hurewicz's theorem [7, § 43, VII, Corollary 3], and hence there exists a compact set $X_\alpha\in \boldsymbol{Q}\setminus \boldsymbol{I}_\alpha$ which we are looking for.

5. Comments

5.1. Let us notice that the spaces C_α we have constructed in Section 4 look much alike Smirnov's "transfinite cubes Q^α" [12] from which some of the combinatorial boundaries of their components (which are finite-dimensional cells) are removed.

5.2. The inductive character of cmp yields that each space C_α in Theorem 2.1 contains for each $\beta\leqslant\alpha$ a closed subspace Z_β such that $\operatorname{cmp} Z_\beta = \beta$; notice that every Z_β contains the compact set $Z_\beta\cap S_\alpha$ with a locally compact complement. This extends the theorem of de Groot and Nishiura [4; Th. 3.1.1] about the existence of spaces with arbitrarily large finite compactness degree.

One can also demonstrate that for each natural n there exists a compact set $X_n\subset Q$ such that (see sec. 4.2, 4.3)

$$n\leqslant \operatorname{cmp} f^{-1}(X_n) < \omega.$$

To see this, consider a compact set $X \subset Q$ with the smallest possible Cantor–Bendixon index such that $\operatorname{cmp} f^{-1}(X) \geqslant \omega$; the existence of compact sets $X_n \subset X$ with the required property follows then easily. The spaces $f^{-1}(X_n)$ have a rather simple structure (see 4.2 (B)), but an essential defect of this construction is that $\operatorname{ind} f^{-1}(X_n) \geqslant \omega$.

5.3. Aarts [0] proved that $\operatorname{cmp}(Q \times I^n) = n$, Q being rational numbers; the reasoning of Aarts yields also that, for any compactum K_α with $\operatorname{ind} K_\alpha = \alpha$, $\operatorname{cmp}(Q \times K_\alpha) = \alpha$. However, neither this method nor the method in [4; Sec. 3.1] yields spaces simultaneously complete and σ-compact and neither of them provides examples with $\operatorname{Ind} \neq \infty$ but $\operatorname{cmp} > \omega$ (cf. Remark 2.2).

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