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A SURVEY OF VARIOUS MODIFICATIONS OF THE NOTIONS OF ABSOLUTE RETRACTS AND ABSOLUTE NEIGHBORHOOD RETRACTS

BY

JACEK KLISOWSKI (LUBLIN)

Our purpose is to expose various modifications of the notions of absolute retracts and absolute neighborhood retracts, originally defined by Borsuk [3], [4] for compact metric spaces and later generalized to other classes of spaces (see [17], [5], and [18]). And in some recent generalizations the usual notion of retraction has been replaced by a more general one. This leads in a natural manner also to other modifications of the original notions. We shall describe some their basic properties and interrelations.

1. Preliminaries. The symbol Q will be used to denote one of the following classes of Hausdorff spaces: class $\mathfrak{M}$ of all metrizable spaces, class $\mathfrak{C}$ of all compact spaces, class of compacta, i.e. compact metrizable spaces. Let $\mathrm{AR}(Q)$ ($\mathrm{ANR}(Q)$) denote the class of absolute (neighborhood) retracts for the class Q (see [17]). Recall that every Q-space can be embedded as a closed subset in some $\mathrm{AR}(Q)$-space.

By a *map* we mean a continuous function. Let X and Y be compacta with metrics ϱ' and ϱ, respectively. Let $\varepsilon > 0$ be a real number. A function $f\colon X \to Y$ is said to be *ε-continuous* if there exists a $\delta > 0$ such that $\varrho'(x, x') < \delta$, $x, x' \in X$, implies $\varrho(f(x), f(x')) < \varepsilon$ (see [25]).

Now let us recall some definitions from [6], [13], and [32]. For the definitions of undefined terms of shape theory used in this paper the reader is referred to the above-mentioned papers.

Let X and Y be closed subsets of $\mathrm{AR}(\mathfrak{M})$-spaces M and N, respectively. A sequence of maps $f_k\colon M \to N$ is called an *S-sequence* from X to Y in M, N (in symbols, $\underline{f} = \{f_k, X, Y\}_{M,N}$) if it satisfies the following two conditions:

(1.1) For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B in N there exists a neighborhood U of A in M such that $f_k | U \simeq f_{k+1} | U$ in V for almost all k.

(1.2) For every neighborhood V of Y in N there exists a neighborhood U of X in M such that $f_k | U \simeq f_{k+1} | U$ in V for almost all k.

By a *W-sequence* (respectively, *P-sequence*) we mean a sequence as the above one which satisfies (1.1) (respectively, (1.2)). In the case where both spaces X and Y are compact, conditions (1.1) and (1.2) are equivalent and then we use the term *fundamental sequence.*

Let X be a closed subset of an ANR$(\mathfrak{M})$-space M. The family $\boldsymbol{U}(X, M)$ of all open neighborhoods of X in M is called a *complete neighborhood system* of X in M. Consider two complete neighborhood systems $\boldsymbol{U}(X, M)$ and $\boldsymbol{U}(Y, N)$. A family $\boldsymbol{f}$ of maps $f\colon U \to V$, where $U \in \boldsymbol{U}(X, M)$ and $V \in \boldsymbol{U}(Y, N)$, is called a *mutation* from $\boldsymbol{U}(X, M)$ to $\boldsymbol{U}(Y, N)$ (in symbols, $\boldsymbol{f}\colon \boldsymbol{U}(X, M) \to \boldsymbol{U}(Y, N)$) if the following three conditions are satisfied:

(1.3) If $f \in \boldsymbol{f}$, $f\colon U \to V$, $U' \subset U$, $U' \in \boldsymbol{U}(X, M)$, $V \subset V' \in \boldsymbol{U}(Y, N)$, and $f'\colon U' \to V'$ is defined by $f'(x) = f(x)$ for $x \in U'$, then $f' \in \boldsymbol{f}$.

(1.4) Every $V \in \boldsymbol{U}(Y, N)$ is the range of some map $f \in \boldsymbol{f}$.

(1.5) If $f_1, f_2 \in \boldsymbol{f}$ and $f_1, f_2\colon U \to V$, then there is a $U' \in \boldsymbol{U}(X, M)$ such that $U' \subset U$ and $f_1 | U' \simeq f_2 | U'$.

By an ANR$(\mathfrak{C})$-*system* we mean an inverse system $\boldsymbol{X} = \{X_a, p_{aa'}, A\}$, where $(A, \leqslant)$ is a closure-finite directed set, i.e. for every $a \in A$ the set of all predecessors of a is finite, and each X_a is an ANR$(\mathfrak{C})$-space. A *map of systems* $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$ consists of an increasing function $f\colon B \to A$ and a collection of maps $f_\beta\colon X_{f(\beta)} \to Y_\beta$ such that $\beta \leqslant \beta'$ implies $f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'}$. Two maps of systems $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{X} \to \boldsymbol{Y}$ are said to be *homotopic*, $\boldsymbol{f} \simeq \boldsymbol{g}$, if for every $\beta \in B$ there exists an $a \in A$, $a \geqslant f(\beta), g(\beta)$, such that $f_\beta p_{f(\beta)a} \simeq g_\beta p_{g(\beta)a}$. An ANR$(\mathfrak{C})$-system $\boldsymbol{X}$ is said to be *associated* with a compact Hausdorff space X if X is homeomorphic to the inverse limit of $\boldsymbol{X}$. Then $p_a\colon X \to X_a$ denotes the natural projection. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be ANR$(\mathfrak{C})$-systems associated with spaces X and Y, respectively. The map of systems $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *associated with a map* $f\colon X \to Y$ if for every $\beta \in B$ we have $f_\beta p_{f(\beta)} \simeq q_\beta f$.

2. Generalized retractions and relations between them. Let X be a closed subset of a metrizable space X'. Consider the following conditions:

HR$(\mathfrak{M})$. There is a map $r\colon X' \to X$ such that $r|X \simeq 1_X$, where $1_X\colon X \to X$ is the identity map.

SR$(\mathfrak{M})$. For some (hence, by [13], p. 52, for every) ANR$(\mathfrak{M})$-space M containing X' as a closed subset there is a mutation $\boldsymbol{r}\colon \boldsymbol{U}(X', M) \to \boldsymbol{U}(X, M)$ such that $r(x) = x$ for $x \in X$ and $r \in \boldsymbol{r}$.

FR$_S(\mathfrak{M})$ (respectively, FR$_W(\mathfrak{M})$, FR$_P(\mathfrak{M})$). For some (hence, by [6], p. 182 and 186, for every) AR$(\mathfrak{M})$-space M containing X' as a closed subset there is an S-sequence (respectively, W-sequence, P-sequence) $\underline{r} = \{r_k, X', X\}_{M,M}$ such that $r_k(x) = x$ for $x \in X$ and $k = 1, 2, \ldots$

$\mathrm{WR}(\mathfrak{M})$. For some (every) $\mathrm{ANR}(\mathfrak{M})$-space M containing X' as a closed subset and for every neighborhood U of X in M there is a map $r: X' \to U$ such that $r(x) = x$ for $x \in X$.

If X satisfies $\mathrm{HR}(\mathfrak{M})$, $\mathrm{SR}(\mathfrak{M})$, $\mathrm{FR_S}(\mathfrak{M})$, $\mathrm{FR_W}(\mathfrak{M})$, $\mathrm{FR_P}(\mathfrak{M})$ or $\mathrm{WR}(\mathfrak{M})$, then X is called a *homotopy retract*, *mutational retract* ([13], p. 53), *S-retract* ([6], p. 186), *W-retract* ([6], p. 183), *P-retract* or *weak retract* of X', respectively.

Remark. It follows from the homotopy extension theorem for mutations ([14], p. 88) that a closed subset X of a metrizable space X' is a mutational retract of X' iff there is a mutation $\boldsymbol{r}: \boldsymbol{U}(X', M) \to \boldsymbol{U}(X, M)$ such that $\boldsymbol{rj} \simeq \boldsymbol{i}$, where the mutations $\boldsymbol{j}: \boldsymbol{U}(X, M) \to \boldsymbol{U}(X', M)$ and $\boldsymbol{i}: \boldsymbol{U}(X, M) \to \boldsymbol{U}(X, M)$ are extensions of the inclusion $j: X \to X'$ and the identity map $i: X \to X$, respectively (for definitions see [13]).

The following diagram shows the relations between the above notions for metrizable spaces (arrow indicates implication):

$$\begin{tikzcd}
 & \mathrm{FR_S}(\mathfrak{M}) \arrow[dl, "\text{by definition}"'] \arrow[dr, "\text{by definition}"] & & & \\
\mathrm{FR_W}(\mathfrak{M}) & & \mathrm{FR_P}(\mathfrak{M}) \arrow[dr, "\text{[13], p. 54}"'] & & \mathrm{HR}(\mathfrak{M}) \arrow[dl, "\text{Theorem (2.4)}"] \\
 & & & \mathrm{SR}(\mathfrak{M}) \arrow[d, "\text{Theorem (2.5)}"] & \\
 & & & \mathrm{WR}(\mathfrak{M}) &
\end{tikzcd}$$

Now, let X be a closed subset of a compact Hausdorff space X'. Consider the following conditions:

$\mathrm{HR}(\mathfrak{C})$. There is a map $r: X' \to X$ such that $r|X \simeq 1_X$.

$\mathrm{SR}(\mathfrak{C})$. For some (hence, by [29], for every pair of) $\mathrm{ANR}(\mathfrak{C})$-systems $\boldsymbol{X}'$ and $\boldsymbol{X}$ associated with X' and X, respectively, there is a map of systems $\boldsymbol{r}: \boldsymbol{X}' \to \boldsymbol{X}$ such that $\boldsymbol{ri} \simeq \boldsymbol{1_X}$, where $\boldsymbol{i}: \boldsymbol{X} \to \boldsymbol{X}'$ is any map of systems associated with the inclusion i and $\boldsymbol{1_X}: \boldsymbol{X} \to \boldsymbol{X}$ is the identity map of systems.

WR($\mathfrak{C}$). For some (every) ANR($\mathfrak{C}$)-space M containing X' and for every neighborhood U of X in M there is a map $r\colon X' \to U$ such that $r(x) = x$ for $x \in X$.

AR($\mathfrak{C}$). For every open covering $\mathfrak{U}$ of X there is a map $r\colon X' \to X$ (called a *$\mathfrak{U}$-retraction*) such that for every $x \in X$ there is a $U \in \mathfrak{U}$ such that $x, r(x) \in U$.

If X satisfies HR($\mathfrak{C}$), SR($\mathfrak{C}$), WR($\mathfrak{C}$) or AR($\mathfrak{C}$), then X is called a *homotopy retract* ([36], [37]), *shape retract* ([29], p. 155), *weak retract* or *approximative retract* [21] of X', respectively.

For compact Hausdorff spaces we have

$$\begin{tikzcd}
\mathrm{HR}(\mathfrak{C}) \arrow[d, "{[29],\ \text{p. } 156}"'] & & \\
\mathrm{SR}(\mathfrak{C}) \arrow[dr, "\text{Theorem (2.5)}"'] & & \mathrm{AR}(\mathfrak{C}) \arrow[dl, "\text{Theorem (2.6)}"] \\
& \mathrm{WR}(\mathfrak{C}) &
\end{tikzcd}$$

Now, let X be a closed subset of a compactum X' (with metric ϱ). Consider the following conditions:

HR. There is a map $r\colon X' \to X$ such that $r|X \simeq 1_X$.

SR. For some (every) AR-space M containing X' there is a fundamental sequence $\underline{r} = \{r_k, X', X\}_{M,M}$ such that $r_k(x) = x$ for $x \in X$ and $k = 1, 2, \ldots$

WR. For some (every) ANR-space M containing X' and for every neighborhood U of X in M there is a map $r\colon X' \to U$ such that $r(x) = x$ for $x \in X$.

AR. For every $\varepsilon > 0$ there is a map $r\colon X' \to X$ (called an *ε-retraction*) such that $\varrho(x, r(x)) < \varepsilon$ for $x \in X$.

SPR. For every $\varepsilon > 0$ there is a neighborhood U of X in X' such that for every $\delta > 0$ there exists a δ-continuous function $r\colon X' \to X$ such that $\varrho(x, r(x)) < \varepsilon$ for $x \in U$.

PR. For every $\varepsilon > 0$ there is an ε-continuous function $r\colon X' \to X$ such that $r(x) = x$ for $x \in X$.

If X satisfies HR, SR, WR, AR, SPR or PR, then X is called a *homotopy retract*, *fundamental retract* [6], *weak retract* ([2], p. 96), *approxima-*

tive retract ([35], p. 20), *strong proximate retract* [25] or *proximate retract* [25] of X', respectively.

Remark. It follows from the homotopy extension theorem for fundamental sequences (see [6], p. 213) that a subcompactum X of a compactum X' is a fundamental retract of X' iff there is a fundamental sequence $\underline{r} = \{r_k, X', X\}_{M,M}$ such that $\underline{r}\underline{j} \simeq \underline{i}$, where fundamental sequences $\underline{j} = \{j_k, X, X'\}_{M,M}$ and $\underline{i} = \{i_k, X, X\}_{M,M}$ are generated by the inclusion $j\colon X \to X'$ and the identity map $i\colon X \to X$, respectively (for definitions see [6]).

For compacta we have

$$\begin{array}{ccccc} \mathrm{AR} & \xrightarrow{[25]} & \mathrm{SPR} & \xrightarrow{[25]} & \mathrm{PR} \\ & & & & \uparrow \big\downarrow {\scriptstyle [2],\ p.\ 102} \\ \mathrm{HR} & \xrightarrow{[29],\ p.\ 156} & \mathrm{SR} & \xrightarrow{[2],\ p.\ 97} & \mathrm{WR} \end{array}$$

Let us notice that for compacta we have the following three propositions:

(2.1) *The conditions* $\mathrm{SR}(\mathfrak{M})$, $\mathrm{SR}(\mathfrak{C})$, *and* SR *are equivalent* (see [13], p. 61, and [29], p. 161).

(2.2) *The conditions* $\mathrm{WR}(\mathfrak{M})$, $\mathrm{WR}(\mathfrak{C})$, *and* WR *are equivalent.*

(2.3) *The conditions* $\mathrm{AR}(\mathfrak{C})$ *and* AR *are equivalent* (see [21]).

From the last remark we obtain the following

(2.4) THEOREM. *If X is a homotopy retract of a metrizable space X', then X is a mutational retract of X'.*

Remark. From now on we shall use the term "shape retract" instead of the terms "mutational retract" and "fundamental retract".

Let us prove the following two theorems:

(2.5) THEOREM. *If X is a shape retract of a Q-space X', then X is a weak retract of X'.*

Proof. This is clear for metrizable spaces.

Now let X' be a compact Hausdorff space. Assume that X' is a subset of a Tychonoff cube I^{Ω}. Let $\boldsymbol{X} = \{X_a, p_{aa'}, A\}$ and $\boldsymbol{X}' = \{X'_a, p'_{aa'}, A\}$ be inclusion $\mathrm{ANR}(\mathfrak{C})$-systems (see [29], p. 159) associated with X and X', respectively. Then all bonding maps and projections are inclusions. By assumption, there exists a map of systems $\boldsymbol{r}\colon \boldsymbol{X}' \to \boldsymbol{X}$ such that $\boldsymbol{ri} \simeq \boldsymbol{1_X}$, where $\boldsymbol{i}\colon \boldsymbol{X} \to \boldsymbol{X}'$ is a map of systems associated with the inclusion i. The map $\boldsymbol{r}$ consists of a collection of maps $r_a\colon X'_{r(a)} \to X_a$, and $\boldsymbol{i}$ consists of a collection of maps $i_a\colon X_{i(a)} \to X'_a$.

Let U be any neighborhood of X in I^{Ω}. Clearly, there is an $a \in A$ such that $X_a \subset U$. Since $ri \simeq 1_X$, there is an $a' \in A$, $a' \geqslant a$, such that $r_a i_{r(a)} p_{ir(a),a'} \simeq p_{aa'}$. We have also $p'_{r(a)} i \simeq i_{r(a)} p_{ir(a)}$ because i is associated with i. Consequently

$$r_a p'_{r(a)} i \simeq r_a i_{r(a)} p_{ir(a)} = r_a i_{r(a)} p_{ir(a),a'} p_{a'} \simeq p_{aa'} p_{a'} = p_a.$$

Since the map $r_a p'_{r(a)}\colon X' \to X_a$ is an extension of the map $r_a p'_{r(a)} i\colon X \to X_a$, and X_a is an absolute neighborhood extensor for compact Hausdorff spaces (by Theorem 8.1 of [17]), we conclude by the homotopy extension theorem ([17], Theorem 28.2) that there is a map $r\colon X' \to X_a$ which is an extension of p_a, i.e. $r(x) = x$ for $x \in X$. This proves that X is a weak retract of X'.

(2.6) THEOREM. *If X is an approximative retract of a compact Hausdorff space X', then X is a weak retract of X'.*

Proof. Assume that X' is a subset of a Tychonoff cube I^{Ω}. Let U be any neighborhood of X in I^{Ω}. There is a family $\mathfrak{V}$ of open convex subsets of U such that $\mathfrak{U} = \{V \cap X : V \in \mathfrak{V}\}$ is a covering of X. By assumption there is a $\mathfrak{U}$-retraction $r\colon X' \to X$. We define a homotopy F of $X \times \langle 0, 1\rangle$ into U by

$$F(x, t) = t \cdot x + (1-t) \cdot r(x) \quad \text{for } x \in X \text{ and } t \in \langle 0, 1\rangle.$$

Since U is an absolute neighborhood extensor for compact Hausdorff spaces (by [17], p. 318), it follows from the homotopy extension theorem that there is a map $s\colon X' \to U$ such that $s(x) = x$ for $x \in X$. This proves that X is a weak retract of X'.

3. Generalized absolute retracts and their properties. In order to make formulation of this section easier we shall use the term *G-retract* to denote one of the terms defined in Section 2. We consider some elementary properties of G-retracts.

(3.1) *If X is a G-retract of X' and X' is a G-retract of X'', then X is a G-retract of X''.*

A closed subset X of a Q-space X' is called a *neighborhood G-retract* of X' if there exists a closed neighborhood V of X in X' such that X is a G-retract of V. Notice that the properties of being a G-retract or a neighborhood G-retract are topological. It is clear that every G-retract of X' is a neighborhood G-retract of X'; the converse statement is not true.

(3.2) *Every (neighborhood) retract of X' is a (neighborhood) G-retract of X'.*

We say that a Q-space X is an *absolute (neighborhood) G-retract for the class Q* and write $X \in \mathrm{AGR}(Q)$ ($X \in \mathrm{ANGR}(Q)$) if, for every Q-space

Y containing X as a closed subset, X is a (neighborhood) G-retract of Y. It is clear that every AGR(Q)-space is an ANGR(Q)-space.

(3.3) *Every* AR(Q)-*space* (*respectively*, ANR(Q)-*space*) *is an* AGR(Q)-*space* (*respectively*, ANGR(Q)-*space*).

(3.4) THEOREM. AGR(Q)-*spaces* (ANGR(Q)-*spaces*) *are the same as* *G-retracts of* AR(Q)-*spaces* (ANR(Q)-*spaces*).

(3.5) COROLLARY. *Every G-retract of an* AGR(Q)-*space* (ANGR(Q)-*space*) *is an* AGR(Q)-*space* (ANGR(Q)-*space*).

Remark. There are also some theorems characterizing AGR(Q)- and ANGR(Q)-spaces by "extendability" (see [36], [37], [6], p. 261, [15], [29], [2], [35], [21], and [25]).

Classes of absolute (neighborhood) homotopy, shape, weak, approximative retracts for the class Q, absolute (neighborhood) S-retracts, W-retracts, P-retracts for metrizable spaces, absolute (neighborhood) proximate and strong proximate retracts for compacta will be denoted by AHR(Q) (ANHR(Q)), ASR(Q) (ANSR(Q)), AWR(Q) (AWNR(Q)), AAR(Q) (AANR(Q)), FAR$_S(\mathfrak{M})$ (FANR$_S(\mathfrak{M})$), FAR$_W(\mathfrak{M})$ (FANR$_W(\mathfrak{M})$), FAR$_P(\mathfrak{M})$ (FANR$_P(\mathfrak{M})$), PAR (PANR), and SPAR (SPANR), respectively. Some proofs of (3.1)-(3.5) can be found in the papers by the following authors who introduced the above notions: Saalfrank [36] and [37], Borsuk [6], Mardešić [29], Godlewski [13], Bogatyĭ [2], Noguchi [35], Kalinin [21], Klee and Yandl [25].

Remark. Let us notice that AWR$(\mathfrak{M})$- and AWNR$(\mathfrak{M})$-spaces are the same as GAR$(\mathfrak{M})$- and ANS$(\mathfrak{M})$-spaces, respectively, introduced by Živanović [39] and [40]. Bourgin [9] introduced δNR's and aNR's which are the same as AWNR$(\mathfrak{C})$- and AANR$(\mathfrak{C})$-spaces, respectively.

Let us give a sketch of proof of the following

(3.6) THEOREM. *Every component of an* ANGR(Q)-*space* X *is open in* X.

Proof. This is known for FANR$_W(\mathfrak{M})$-spaces ([6], p. 193) and can easily be proved for AWNR(Q)-spaces. Hence, by the results of Section 2, we obtain the thesis.

(3.7) COROLLARY. *Every compact* ANGR(Q)-*space has a finite number of components.*

(3.8) COROLLARY. *Every component of an* ANGR(Q)-*space is again an* ANGR(Q)-*space.*

Let us mention one modification of the notion of a neighborhood approximative retract. A closed subset X of a compact Hausdorff space X' is called a ***neighborhood approximative retract in the sense of Clapp*** (see [10] and [21]) of X' if for every covering $\mathfrak{U}$ of X there are a neighborhood

V of X in X' and a $\mathfrak{U}$-retraction $r\colon V \to X$. It is clear that every neighborhood approximative retract of X' is a neighborhood approximative retract in the sense of Clapp of X' and every AANR$(\mathfrak{C})$-space is an absolute neighborhood approximative retract in the sense of Clapp (AANR$_C(\mathfrak{C})$-space). It is known that AANR$_C$-spaces are the same as NE-sets introduced by Borsuk [7].

Let us notice that the results of Section 2 imply analogous relations between neighborhood generalized retracts and between absolute (neighborhood) generalized retracts. Some other relations will be studied in the next section.

4. The relations between absolute (neighborhood) generalized retracts.

(4.1) THEOREM. *Every* AAR$(\mathfrak{C})$*-space* (AANR$(\mathfrak{C})$*-space*) *is an* ASR$(\mathfrak{C})$*-space* (ANSR$(\mathfrak{C})$*-space*) ([21], [12], see also [2]).

One can easily show the following (compare with [8], p. 163, Epilogue by Iu. M. Smirnov)

(4.2) PROPOSITION. *If X is a Q-space, then the following conditions are equivalent*:

(1) $X \in$ AWR(Q).

(2) *X is contractible in each of its neighborhoods in some* (*every*) ANR(Q)*-space containing X as a closed subset.*

(3) *Every neighborhood U of X in some* (*every*) ANR (Q)*-space M containing X as a closed subset contains a neighborhood U_0 of X in M which is contractible in U.*

(4.3) THEOREM. *A Q-space X is an* ASR(Q)*-space iff X is an* AWR(Q)*-space.*

Proof. It follows from Theorem (2.5) that every ASR(Q)-space is an AWR(Q)-space.

Now assume that $X \in$ AWR$(\mathfrak{M})$. Then X is a weak retract of some AR$(\mathfrak{M})$-space M. Let $\boldsymbol{r}\colon \boldsymbol{U}(M, M) \to \boldsymbol{U}(X, M)$ be a family of all maps $r\colon M \to U$ such that $U \in \boldsymbol{U}(X, M)$ and $r(x) = x$ for $x \in X$. It is clear that $\boldsymbol{r}$ is a mutation. We conclude from Theorem (3.4) that $X \in$ ASR$(\mathfrak{M})$.

Now assume that $X \in$ AWR$(\mathfrak{C})$. Let $\boldsymbol{X} = \{X_a, p_{aa'}, A\}$ be an inclusion ANR$(\mathfrak{C})$-system associated with X. It follows from Proposition (4.2) that for every $a \in A$ there is an $a' \in A$, $a' \geqslant a$, such that $X_{a'}$ is contractible in X_a, i.e. $p_{aa'}$ is null-homotopic. Hence from Theorem 5 of [29] we infer that $X \in$ ASR$(\mathfrak{C})$.

For compacta this is also proved in [8] (Theorem 27.1) and in [39].

The following theorem is well known:

(4.4) THEOREM. *A Q-space is an* ASR(Q)*-space iff its shape* (*in the sense of* [34]) *is trivial* ([6], p. 257, [14], p. 90, and [29], p. 156).

There are also some other characterizations of $\mathrm{ASR}(\mathfrak{C})$-spaces [22], ASR-spaces ([19], [20], [30], [27]), and ANSR-spaces ([6], p. 254, [31]).

One can easily prove that

(4.5) $\mathrm{AHR}(Q)$-*spaces are the same as contractible* Q-*spaces.*

Now let us give some propositions connecting $\mathrm{AWR}(Q)$-spaces with other absolute generalized retracts.

(4.6) PROPOSITION. *A* Q-*space* X *is an* $\mathrm{AR}(Q)$-*space iff* X *is an* $\mathrm{AWR}(Q)$- *and* $\mathrm{ANR}(Q)$-*space.*

The metrizable case is proved in [39] (Theorem 2.10). By similar arguments one can easily prove other cases and the following

(4.7) PROPOSITION. *A* Q-*space* X *is an* $\mathrm{AHR}(Q)$-*space iff* X *is an* $\mathrm{AWR}(Q)$- *and* $\mathrm{ANHR}(Q)$-*space.*

(4.8) PROPOSITION. *A compact Hausdorff space* X *is an* $\mathrm{AAR}(\mathfrak{C})$-*space iff* X *is an* $\mathrm{AWR}(\mathfrak{C})$- *and* $\mathrm{AANR}_{\mathrm{C}}(\mathfrak{C})$-*space* ([2], p. 98, [21], p. 731).

(4.9) PROPOSITION. *A compact Hausdorff space* X *is an* $\mathrm{AANR}(\mathfrak{C})$-*space iff* X *is an* $\mathrm{AWNR}(\mathfrak{C})$- *and* $\mathrm{AANR}_{\mathrm{C}}(\mathfrak{C})$-*space* (for the proof for compacta see [2], p. 99).

(4.10) PROPOSITION. *A compactum* X *is an* SPAR-*space iff* X *is an* AWR- *and* SPANR-*space.*

Proof. It is clear that every SPAR-space is an AWR- and SPANR-space.

Now assume that X is an AWR- and SPANR-space contained in some AR-space M (with metric ϱ). There are a neighborhood W of X in M such that X is a strong proximate retract of W and a map $s\colon M \to W$ such that $s(x) = x$ for $x \in X$.

Let $\varepsilon > 0$ be any real number. Then there is a neighborhood U of X in W such that for every $\delta > 0$ there is a δ-continuous function $r\colon W \to X$ such that $\varrho(x, r(x)) < \frac{1}{2}\varepsilon$ for $x \in U$. Observe that a set

$$V = s^{-1}(U) \cap \{x \in M\colon \varrho(x, s(x)) < \tfrac{1}{2}\varepsilon\}$$

is a neighborhood of X in M. For a given $\delta > 0$ let $r\colon W \to X$ be a δ-continuous function such that $\varrho(x, r(x)) < \frac{1}{2}\varepsilon$ for $x \in U$. Then the function $rs\colon M \to X$ is δ-continuous and

$$\varrho(rs(x), x) \leqslant \varrho(rs(x), s(x)) + \varrho(s(x), x) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

for $x \in V$. This proves that X is a strong proximate retract of M and by Theorem (3.4) it is an SPAR-space.

(4.11) PROPOSITION. *If a metrizable space* X *is an* $\mathrm{AWR}(\mathfrak{M})$- *and* $\mathrm{FANR}_{\mathrm{S}}(\mathfrak{M})$-*space* ($\mathrm{FANR}_{\mathrm{P}}(\mathfrak{M})$-*space,* $\mathrm{FANR}_{\mathrm{W}}(\mathfrak{M})$-*space*)*, then* X *is an* $\mathrm{FAR}_{\mathrm{S}}(\mathfrak{M})$-*space* ($\mathrm{FAR}_{\mathrm{P}}(\mathfrak{M})$-*space,* $\mathrm{FAR}_{\mathrm{W}}(\mathfrak{M})$-*space*).

Proof. Assume that X is a closed subset of some $\mathrm{AR}(\mathfrak{M})$-space M. Let $\underline{r} = \{r_k, V, X\}_{M,M}$ be an S-retraction (P-retraction, W-retraction), where V is a closed neighborhood of X in M. Since $X \in \mathrm{AWR}(\mathfrak{M})$, there is a map $s\colon M \to V$ such that $s(x) = x$ for $x \in X$. It remains to prove that if $\underline{r}$ is a P-sequence (W-sequence), then $\underline{s} = \{r_k s, M, X\}_{M,M}$ is a P-sequence (W-sequence).

First assume that $\underline{r}$ is a P-sequence. Let U be any neighborhood of X in M. There is a neighborhood V' of V in M such that $r_k | V' \simeq r_{k+1} | V'$ in U for almost all k. Then $r_k s \simeq r_{k+1} s$ in U for almost all k. This proves that $\underline{s}$ is a P-sequence.

Now assume that $\underline{r}$ is a W-sequence. Let A be any compactum in M. Then $s(A)$ is a compactum in V. Let U be any neighborhood of B in M, where B is a compactum in X associated with $s(A)$ (see the condition $\mathrm{FR_W}(\mathfrak{M})$). Then there is a neighborhood V' of $s(A)$ in M such that $r_k | V' \simeq r_{k+1} | V'$ in U for almost all k. Let $U' = s^{-1}(V')$. Then U' is a neighborhood of A in M such that $r_k s | U' \simeq r_{k+1} s | U'$ in U for almost all k. This proves that $\underline{s}$ is a W-sequence.

Clearly, for $\mathrm{FAR_S}(\mathfrak{M})$- and $\mathrm{FAR_P}(\mathfrak{M})$-spaces, the converse statement is also true.

5. Examples. Let us give some examples which show that the converses of the statements in Section 2 are not true.

(5.1) Let us denote by X_n, $n = 1, 2, \ldots$, the closed half-line in the plane E^2 with the end-point $(0, 0)$ and containing the point $(n, 1)$, and by X_0 the half-line with the end-point $(0, 0)$ and containing the point $(1, 0)$. Let $X = \bigcup_{n=0}^{\infty} X_n$. Then X is an $\mathrm{AHR}(\mathfrak{M})$-space which is not an $\mathrm{FANR_P}(\mathfrak{M})$-space (see [16]). One can easily see that X is an $\mathrm{FAR_W}(\mathfrak{M})$-space.

However, not all relations between $\mathrm{FAR_S}(\mathfrak{M})$-, $\mathrm{FAR_P}(\mathfrak{M})$- and $\mathrm{FAR_W}(\mathfrak{M})$-spaces are known.

(5.2) Let X be any infinite 0-dimensional compactum. Then X is an $\mathrm{AANR_C}$-space ([2], p. 93) but, by Corollary (3.7), it is not even an AWNR-space.

(5.3) Let X_n, $n = 1, 2, \ldots$, be a decreasing sequence of compacta such that X_{n+1} is a retract of X_n for all n, but $X = \bigcap_{n=1}^{\infty} X_n$ is a compactum which is not a shape retract of X_1 (see [11]). It is clear that X is a weak retract of X_1.

We do not know if there exists an $\mathrm{AWNR}(Q)$-space which is not an $\mathrm{ANSR}(Q)$-space (for compacta it is the question of Živanović [39] and Bogatyĭ [2]). **(P 1244)**

(5.4) Let X be a contractible continuum which has no fixed point property [24]. Then X is an AHR-space but it is not an SPAR-space because, by the results of [25], every SPAR-space has fixed point property. Hence, by Proposition (4.10), X is not an SPANR-space.

(5.5) Let X denote the closure of the curve in the plane E^2 whose equation is $y = \sin \pi/x$ for $0 < x \leqslant 1$, and let L denote the segment with the end-points $(-1, 0)$ and $(0, 0)$. Then one can easily see that $Y = X \cup L$ is an SPAR-space which is not an AAR-space. Hence, by Proposition (4.8), Y is not an AANR_C-space. This example gives a negative answer to the problem of [25].

Let us notice that X is an AAR-space which is not an AHR-space, and hence, by Proposition (4.7), X is not even an ANHR-space.

6. A short review of other modified AR- and ANR-spaces. First let us mention the notion of *absolute* (*neighborhood*) *homology retracts* introduced by Lefschetz [28]. Sher [38] has studied *absolute* (*neighborhood*) *proper retracts*, where retractions were restricted by some conditions. Ball [1] transferred the notions of AR- and ANR-spaces to proper shape theory which is a remarkable extension of shape theory from compacta onto locally compact metrizable spaces. Kozlowski and Segal [26] extended the notions of ANSR-spaces to paracompacta.

Instead of continuous retractions one can consider more general almost continuous and weakly continuous retractions. This leads to modifications of AR- and AAR-spaces but, as has been proved by Kellum [23], these classes are the same as the class of AAR-spaces. It is not known if an analogous statement is true for AANR-spaces. (**P 1245**)

Finally, let us mention another notion. A closed subset X of a space X' is called an *approximate retract* of X' if for every neighborhood U of X in X' there is a retract R of X' such that $X \subset R \subset U$. This leads to the notion of *absolute approximate retracts* [33]. Notice that here analogues of (3.1) and (3.4) are not true. It is not known if there exists an absolute approximate retract which is not an AR-space. (**P 1246**)

Added in proof. K. Tsuda (*On AWNR-spaces in shape topology*, Mathematica Japonica 22 (1977), p. 471-478) has given a partial answer to P 1244.

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INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSITY
LUBLIN

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