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ON THREE PROBLEMS OF FRANKLIN AND WALLACE CONCERNING PARTIALLY ORDERED SPACES*

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1. Introduction. In what follows a *partially ordered space* is a topological space X endowed with a partial order which, regarded as a subset of $X \times X$, is closed. If the space X is endowed with a closed, reflexive and transitive relation, then it is called a *quasi ordered space.* The symbol 2^X denotes the space of closed subsets of X with the Vietoris topology [4]. That is, if $\{U_1, \ldots, U_n\}$ is a finite collection of open subsets of X, then $\langle U_1, \ldots, U_n \rangle$ denotes the set of all $A \in 2^X$ such that $A \subset U_1 \cup \ldots \cup U_n$ and $A \cap U_i$ is non-empty for each $i = 1, \ldots, n$. The family of all such $\langle U_1, \ldots, U_n \rangle$ is a base for the Vietoris topology.

If X is a quasi-ordered space, then $\mathscr{M}(X)$ denotes the family of all maximal chains of X. It is known [6] that $\mathscr{M}(X)$ is a subset of 2^X. The set of all closed chains of X is denoted by $\mathscr{C}(X)$. We let $\mathrm{Max}(X)$ (resp., $\mathrm{Min}(X)$) be the set of maximal (resp., minimal) elements of X. If X is compact, then $\mathrm{Max}(X)$ and $\mathrm{Min}(X)$ are non-empty [6]. If R is any relation on X we follow the standard terminology:

$$Rx = \{y \in X : (y, x) \in R\}, \qquad xR = \{y \in X : (x, y) \in R\}$$

for each $x \in X$. However, in the case of a quasi-order Q we shall also write $x \leqslant y$ when $(x, y) \in Q$ as well as $L(x) = Qx$ and $M(x) = xQ$, for each $x \in X$. If $A \subset X$, then

$$L(A) = \bigcup\{L(x) : x \in A\}, \qquad M(A) = \bigcup\{M(x) : x \in A\}.$$

Two quasi-orders P and Q on X are said to be *chain equivalent* if $P \cup P^{-1} = Q \cup Q^{-1}$. Finally, if R is a reflexive relation on X, then Σ denotes the subfamily of 2^X each of whose members contains an R-least element.

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Recently Franklin and Wallace [3] have asked three questions on relations in topological spaces which we are able to answer in whole or in part.

P 555. *If X is a compact quasi-ordered Hausdorff space, is the set of maximal members of Σ a closed subset of 2^X?*

P 556. *If X is a compact partially ordered space, is $\mathscr{M}(X)$ a closed subset of 2^X?*

P 557. *If X is a compact quasi-ordered Hausdorff space with quasi-order Q, under what conditions does Q contain a closed partial order which is chain equivalent to Q?*

2. Problem 555. The theorem which follows shows that the answer to this question is in the negative, even if the relation is a closed partial order. In fact, in this setting a complete answer is easy to establish.

THEOREM 1. *Let X be a compact partially ordered space and let Σ_M denote the set of maximal members of Σ. Then Σ_M is a closed subset of 2^X if and only if $\operatorname{Min}(X)$ is closed and the mapping $\varphi : \operatorname{Min}(X) \to 2^X$ defined by $\varphi(x) = M(x)$ is continuous.*

Proof. We note first that

$$\Sigma_M = \{M(x) : x \in \operatorname{Min}(X)\} = \varphi\big(\operatorname{Min}(X)\big).$$

If $\operatorname{Min}(X)$ is closed and φ is continuous, then Σ_M is compact and hence closed in 2^X. Conversely, if Σ_M is closed, suppose e_a is a net in $\operatorname{Min}(X)$ which converges to $e \in X$; then the net $M(e_a)$ in Σ_M has a cluster point $M(e_1)$, where $e_1 \in \operatorname{Min}(X)$. Since the partial order has a closed graph, it is easy to see that $e_1 = e$. This proves that $\operatorname{Min}(X)$ is closed and that φ is continuous.

3. Problem 556. For this problem it is convenient to use the following result of Nachbin [5]:

THEOREM 2. *Let X be a compact partially ordered space and suppose that F_0 and F_1 are closed subsets of X such that x_1 non $\leqslant x_0$ whenever $x_0 \in F_0$ and $x_1 \in F_1$. Then there exist disjoint open sets U_0 and U_1 such that $F_0 \subset U_0 = L(U_0)$ and $F_1 \subset U_1 = M(U_1)$.*

LEMMA 3.1. *If X is a partially ordered space, then $\mathscr{C}(X)$ is a closed subset of 2^X.*

Proof. If $A \in 2^X - \mathscr{C}(X)$, then A contains elements a_1 and a_2 which are not comparable. Since the partial order has a closed graph there exist open sets U_1 and U_2 containing a_1 and a_2, respectively, and such that no element of U_1 is comparable with any element of U_2. It follows that $A \in \langle U_1, U_2, X\rangle$ but that no chain is a member of $\langle U_1, U_2, X\rangle$, and hence that $\mathscr{C}(X)$ is closed.

A partially ordered set S is *order-dense* provided whenever a and b are members of S and $a < b$, then there exists $c \in S$ with $a < c < b$.

THEOREM 3. *Let X be a compact, order-dense partially ordered space. Then $\mathscr{M}(X)$ is a closed subset of 2^X if and only if* $\mathrm{Max}(X)$ *and* $\mathrm{Min}(X)$ *are closed sets.*

Proof. If $\mathrm{Max}(X)$ is not closed, then there exists a net e_a in $\mathrm{Max}(X)$ with $e_a \to a \in X - \mathrm{Max}(X)$. For each a there exists $M_a \in \mathscr{M}(X)$ with $e_a = \sup M_a$. By Lemma 3.1 we know that $\mathscr{C}(X)$ is closed in 2^X, and, since 2^X is compact, the net M_a has a subnet converging to a closed chain C. Since the partial order has a closed graph, it follows that $\sup C = a$ and hence C is a member of the closure of $\mathscr{M}(X)$ but not of $\mathscr{M}(X)$. A dual argument applies if $\mathrm{Min}(X)$ is not closed.

Conversely, suppose that $\mathrm{Max}(X)$ and $\mathrm{Min}(X)$ are both closed sets. In view of Lemma 3.1 it suffices to show that if $C \in \mathscr{C}(X) - \mathscr{M}(X)$, then C lies in an open set of 2^X which is disjoint from $\mathscr{M}(X)$. Since C is compact, it has a supremum and an infimum which it contains. Since C is not maximal, we distinguish three cases: either $\sup C \in X - \mathrm{Max}(X)$, or $\inf C \in X - \mathrm{Min}(X)$, or $C = C_0 \cup C_1$, where C_0 and C_1 are closed chains and there exists an element which is strictly between $\sup C_0$ and $\inf C_1$.

In the first case we note that C and $\mathrm{Max}(X)$ are closed sets satisfying the hypotheses of Theorem 2 and hence there is an open set U with

$$C \subset U = L(U) \subset X - \mathrm{Max}(X).$$

Accordingly, $C \in \langle U \rangle$, but since every maximal chain meets $\mathrm{Max}(X)$, it follows that $\langle U \rangle$ and $\mathscr{M}(X)$ are disjoint. A dual argument applies in the second case.

In the third case we again apply Theorem 2. There exist disjoint open sets U_0 and U_1 such that $C_0 \subset U_0 = L(U_0)$ and $C_1 \subset U_1 = M(U_1)$ so that $C \in \langle U_0, U_1 \rangle$. But no member of $\mathscr{M}(X)$ can lie in $\langle U_0, U_1 \rangle$ since the maximal chains of a compact, order-dense partially ordered space are connected [6].

The hypothesis of order-density is essential to Theorem 3. For in the Hilbert cube I^ω let

$$T_n = \{t : t_m = 0 \text{ if } m \neq n \text{ and } 0 \leqslant t_n \leqslant 2^{-n}\}$$

and let $S = \bigcup_{n=1}^{\infty} \{T_n\} \cup \{1\}$. We give S the coordinatewise partial order inherited from I^ω: $x \leqslant y$ if and only if $x_n \leqslant y_n$ for each $n = 1, 2, \ldots$, Then S is a compact partially ordered space and $\mathrm{Max}(S)$ and $\mathrm{Min}(S)$ are singletons. The maximal chains of S are all of the form $T_n \cup \{1\}$ and it is a routine exercise to verify that $\{0, 1\}$ lies in the closure of $\mathscr{M}(S)$.

As a matter of fact S is actually a lattice, but the join operation is not continuous. If it is assumed that S is a topological lattice; then no assumptions of order-density are necessary in order to guarantee that $\mathscr{M}(S)$ is closed.

THEOREM 4. *If S is a compact topological lattice, then $\mathscr{M}(S)$ is a closed subset of 2^S.*

Proof. By Lemma 3.1 it suffices to show that if $C \in \mathscr{C}(S) - \mathscr{M}(S)$, then C has a neighborhood disjoint from $\mathscr{M}(S)$. As in the proof of Theorem 3 we distinguish three cases.

If $\sup C < 1$, then C is contained in an open set U such that $U = L(U)$ and $1 \in S - U$, by Theorem 2. Thus $C \in \langle U \rangle$ and no maximal chain can be a member of $\langle U \rangle$. The case $0 < \inf C$ follows in the same way. It remains to consider the case where $C = C_0 \cup C_1$, where C_0 and C_1 are members of $\mathscr{C}(S)$ and there exists $t \in S$ such that $\sup C_0 < t < \inf C_1$. If each neighborhood of C meets $\mathscr{M}(S)$, then there exists a net M_α in $\mathscr{M}(S)$ such that $M_\alpha \to C$ in the space 2^S. By Theorem 2 there are disjoint open sets U_0 and U_1 such that $C_0 \subset U_0 = L(U_0)$ and $C_1 \subset U_1 = M(U_1)$, and hence the net M_α is eventually in $\langle U_0, U_1 \rangle$. That is, eventually $M_\alpha = M_{\alpha,0} \cup M_{\alpha,1}$, where $M_{\alpha,0} \to C_0$ and $M_{\alpha,1} \to C_1$. Since the partial order is closed, a simple argument shows that $\sup M_{\alpha,0} \to \sup C_0$ and $\inf M_{\alpha,1} \to \inf C_1$. Consequently, if we define

$$t_\alpha = (\sup M_{\alpha,0} \vee t) \wedge \inf M_{\alpha,1},$$

then t_α converges to $(\sup C_0 \vee t) \wedge \inf C_1 = t$. In particular, t_α is eventually in the complement of M_α. But it is clear from the definition of t_α that

$$\sup M_{\alpha,0} \leqslant t_\alpha \leqslant \inf M_{\alpha,1},$$

so that $M_\alpha \cup \{t_\alpha\}$ is a chain, contrary to the maximality of M_α.

4. Problem 557. The following result is contained in the dissertation of Franklin [2]:

THEOREM 5. *Let X be a compact Hausdorff space. Then the relation $R \subset X \times X$ is closed if and only if the set function $x \to xR$ is upper semi-continuous and has closed point-images.*

If X is a quasi ordered space, we write $E(x) = L(x) \cap M(x)$ for each $x \in X$.

THEOREM 6. *Let X be a compact quasi ordered Hausdorff space with quasi order Q. Then Q contains a closed chain equivalent partial order P if and only if for each $x \in X$ there exists a simple order Γ_x on $E(x)$ such that*

(i) *$\Gamma_x = \Gamma_y$ if $y \in E(x)$,*

(ii) *the set function $x \to (xQ - Qx) \cup x\Gamma_x$ is upper semi-continuous and has closed point-images.*

Proof. If (i) and (ii) are satisfied, let $(x, y) \in P$ if and only if $y \in (xQ - Qx) \cup x\Gamma_x$. It is routine to verify that P is a partial order, $P \subset Q$ and P and Q are chain equivalent. That P has a closed graph follows from Theorem 5. Conversely, the chain equivalence of P and Q implies that each set $E(x)$ is a P-chain; letting $\Gamma_x = P \cap (E(x) \times E(x))$ we have (i), and (ii) follows from Theorem 5.

Under certain conditions we can assert the *uniqueness* of the partial order P. First we establish

LEMMA 7.1. *Let Q be a closed quasi order on the compact space X and suppose Q contains a closed chain equivalent partial order P. Suppose that $xQ \cup Qx$ is connected, for each $x \in X$. Then*

(i) *$E(x)$ is a (possibly degenerate) arc* (1) *which is also a P-chain,*

(ii) *if m_x and l_x are the P-maximal and P-minimal elements, respectively, of $E(x)$, then m_x and l_x are the endpoints of $E(x)$,*

(iii) *if $xQ - Qx$ is non-empty (resp., $Qx - xQ$ is non-empty), then $m_x = \overline{(xQ - Qx)} \cap E(x)$ (resp., $l_x = \overline{(Qx - xQ)} \cap E(x)$).*

Proof. If $x \in X$, then $E(x)$ is by Theorem 6 a P-chain and $E(x)$ is closed since $E(x) = xQ \cap Qx$. Note that

$$(*) \qquad xQ \cup Qx = m_x P \cup E(x) \cup P l_x$$

which is a connected set by hypothesis. If $E(x)$ is not degenerate, then $m_x P$ and $P l_x$ are disjoint closed sets and hence $E(x)$ is connected. Since $E(x)$ is a continuum and a chain, it is an arc. The assertion (ii) is clear. To verify (iii) we may assume that $x = m_x$; then $xQ - Qx = xP - x$, so that by $(*)$ and the connectedness of $xQ \cup Qx$ we may conclude

$$\overline{xQ - Qx} = \overline{xP - x} = xP.$$

Therefore

$$\overline{(xQ - Qx)} \cap E(x) = x = m_x,$$

and the statement for l_x follows by a dual argument.

THEOREM 7. *Let Q be a closed quasi-order on the compact space X and suppose that $xQ \cup Qx$ is connected and that $E(x) \neq xQ \cup Qx$ for each $x \in X$. If Q contains a closed chain equivalent partial order P, then P is unique.*

Proof. For if P and P' are distinct closed partial orders which are contained in Q and are chain equivalent to Q, suppose $(x, y) \in P - P'$. Then $(y, x) \in P' \subset Q$ and hence $y \in E(x)$. By Lemma 7.1 the arc $E(x)$ is a chain relative to both P and P'. It follows from (iii) of Lemma 7.1 that $P \cap (E(x) \times E(x)) = P' \cap (E(x) \times E(x))$, a contradiction.

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(1) An *arc* is a continuum with exactly two non-cutpoints.

If the spaces under consideration are taken to be metric spaces, then the notion of a radially convex metric sheds light on the existence of closed chain equivalent partial orders. If X is a space with metric ϱ and P is a partial order on X, then ϱ is *radially convex* (*with respect to* P) provided whenever $x \leqslant y \leqslant z$ in X it follows that $\varrho(x, y) + \varrho(y, z) = \varrho(x, z)$. The following basic theorem on such metrics is due to Carruth [1].

THEOREM 8. *Every compact metric partially ordered space admits a radially convex metric.*

THEOREM 9. *Let Q be a closed quasi-order on the compact metric space X, and let P be a partial order on X such that $P \subset Q$, P is chain equivalent to Q and Px is closed for each $x \in X$. Suppose in addition that* $\mathrm{Min}(X, P)$, *the set of P-minimal elements of X, is a closed set. Then P is closed if and only if X admits a metric which is radially convex with respect to P.*

Proof. If P is closed then the existence of a radially convex metric follows at once from Theorem 8. Conversely, suppose X admits a metric ϱ which is radially convex with respect to P. Let x_a and y_a be nets in X such that $x_a \to x$, $y_a \to y$ and $(x_a, y_a) \in P$ for each a. It is sufficient to show that $(x, y) \in P$. Since P is chain equivalent to Q and Q is closed, we have $(x, y) \in Q$ or $(y, x) \in Q$. Since the sets Px_a are closed, there exists $n_a \in \mathrm{Min}(X, P) \cap Px_a$, and since $\mathrm{Min}(X, P)$ is closed, the net n_a has a cluster point $n \in \mathrm{Min}(X, P)$. Since Q is closed, we infer that $(n, x) \in Q$ and $(n, y) \in Q$. Now

$$\varrho(n, x) = \lim \varrho(n_a, x_a) \leqslant \lim \varrho(n_a, y_a) = \varrho(n, y)$$

since ϱ is radially convex, and hence $(x, y) \in P$.

We give one more theorem on the existence of closed chain equivalent partial orders which is independent of what has gone before,

THEOREM 10. *Let Q be a closed quasi-order on the compact space X. let $S = \{x \in X : E(x) \neq x\}$ and suppose the following three conditions are satisfied:*

(i) *$X - S$ is dense in X,*

(ii) *the mapping $x \to (xQ \cup Qx)$ is continuous,*

(iii) *if x and y are distinct elements of X and $(x, y) \in Q$, then there exist disjoint neighborhoods U and V of x and y, respectively, such that either*

(iiia) $(Q \cup Q^{-1}) \cap (U \times V) \subset Q$ or

(iiib) $(Q \cup Q^{-1}) \cap (U \times V) \subset Q^{-1}$.

Then Q contains a closed chain equivalent partial order.

Proof. Define

$$P = Q \cap \{(x, y)\,;\ \text{(iiia) holds}\}.$$

Since (iiia) is satisfied vacuously on the diagonal of X, it is clear that P is reflexive. Moreover, if (iiib) holds for (x, y), then $(y, x) \in P$ and hence $P \cup P^{-1} = Q \cup Q^{-1}$. To prove that P is asymmetric, suppose (x, y) and (y, x) are members of P with $x \neq y$. Then disjoint open sets U and V may be chosen so that $x \in U$, $y \in V$ and

$$(Q \cup Q^{-1}) \cap (U \times V) \subset Q \cap Q^{-1}.$$

Moreover, by (ii) the open sets U and V can be chosen so that if $x' \in U$, then there exists

$$y' \in V \cap (x'Q \cup Qx')$$

and hence

$$(x', y') \in (U \times V) \cap (Q \cup Q^{-1}) \subset Q \cap Q^{-1},$$

i.e., $y' \in E(x')$. But by (i) we may choose $x' \in U$ such that $E(x') = x'$. This contradicts the assumption that U and V are disjoint.

To see that P is transitive, suppose $(x, y) \in P$ and $(y, z) \in P$, where x, y and z are all distinct. Then there are neighborhoods U, V and W of x, y and z, respectively, which are mutually disjoint and are such that

$$(Q \cup Q^{-1}) \cap (U \times V) \subset Q, \qquad (Q \cup Q^{-1}) \cap (V \times W) \subset Q.$$

By the transitivity of Q, we infer

$$(Q \cup Q^{-1}) \cap (U \times W) \subset Q$$

and hence $(x, z) \in P$.

It remains to prove that P is closed. Let (x_a, y_a) be a net in P with $(x_a, y_a) \to (x, y)$; then $(x, y) \in Q$ since Q is closed and we may assume $x \neq y$. Choose U and V according to (iii). Eventually $(x_a, y_a) \in U \times V$ and since U and V are disjoint, we have $(x_a, y_a) \in P - P^{-1}$. Thus $(Q \cup Q^{-1}) \cap$ $\cap (U \times V)$ cannot be contained in Q^{-1} and there exists $x' \in U - S$ and $y' \in V$ with $(x', y') \in Q - Q^{-1}$. It follows that (iiia) must occur and hence $(x, y) \in P$. The proof is complete.

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