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ON THE SYMMETRIC DERIVATIVE

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In the theory of real functions there is a well-known theorem of Banach-Mazurkiewicz (see [1] and [3]), which deals with the metric space C of all real continuous functions on the interval $\langle 0, 1\rangle$ with the metric

$$\varrho(f, g) = \max_{x\in\langle 0,1\rangle}\{|f(x)-g(x)|\}, \quad f, g\in C,$$

and explains the structure of the set of all functions $f\in C$ such that for every $x\in(0, 1)$ there is

$$\limsup_{h\to 0}\big(f(x+h)-f(x)\big)/h = +\infty$$

and

$$\liminf_{h\to 0}\big(f(x+h)-f(x)\big)/h = -\infty.$$

In the present paper we shall prove an analogue of this theorem for the symmetric derivative.

THEOREM. *Let M be the set of all functions $f\in C$ with the following properties: for each $x\in(0, 1)$ there is*

$$\limsup_{h\to 0}\big(f(x+h)-f(x-h)\big)/2h = +\infty,$$

$$\liminf_{h\to 0}\big(f(x+h)-f(x-h)\big)/2h = -\infty.$$

Then the set $N = C-M$ is of the first category in C.

Let

$$\Phi_f(x, h) = \big(f(x+h)-f(x-h)\big)/2h.$$

In the proof of the theorem we shall use the following lemma:

LEMMA. *There is a function $\varphi\in C$ such that*

$$\limsup_{h\to 0}\Phi_\varphi(x, h) = +\infty$$

holds for each $x\in(0, 1)$.

Function φ with the required property is constructed in [2].

Proof of the theorem. Let $N = N_1 \cup N_2$, where

$$N_1 = \{f \in C\colon \text{ there exists } x \in (0,1) \text{ such that } \limsup_{h\to 0} \Phi_f(x,h) < +\infty\},$$

$$N_2 = \{f \in C\colon \text{ there exists } x \in (0,1) \text{ such that } \liminf_{h\to 0} \Phi_f(x,h) > -\infty\}.$$

It is sufficient to prove that the two sets N_1 and N_2 are both of the first category in C. We shall do it first for the set N_1. Namely, we shall prove that the complement of N_1 is dense in C and the set N_1 is of the type F_σ (see [4], p. 88).

Let $\varepsilon > 0$ and let $K(p, \varepsilon) = \{f \in C\colon \varrho(f, p) < \varepsilon\}$, where p is a polynomial. We show that $K(p, \varepsilon) \cap (C - N_1) \neq \emptyset$. Every function of the form $p + \eta\varphi$, $\eta > 0$, where φ is a function the existence of which is guaranteed by the lemma, belongs to $C - N_1$. In fact, if the polynomial p satisfies the Lipschitz's condition with a constant L (i.e., $|p(x) - p(x')| \leqslant L|x - x'|$, $x, x' \in \langle 0, 1\rangle$), then for each $x \in (0,1)$ there is

$$\Phi_{p+\eta\varphi}(x,h) = \Phi_p(x,h) + \eta\Phi_\varphi(x,h) \geqslant -L + \eta\Phi_\varphi(x,h),$$

whence

$$\limsup_{h\to 0} \Phi_{p+\eta\varphi}(x,h) = +\infty.$$

If we put

$$\eta = \varepsilon/2\|\varphi\| \qquad (\|\varphi\| = \max_{x\in\langle 0,1\rangle}\{|\varphi(x)|\}),$$

then, obviously, $p + \eta\varphi \in K(p, \varepsilon)$.

Let $F_n = \{f \in C\colon$ there exists $x \in \langle 1/n, 1 - 1/n\rangle$ such that if $0 < |h| < 1/n$, then $\Phi_f(x,h) \leqslant n\}$ for $n = 2, 3, \ldots$ Since $N_1 = \bigcup_{n=2}^{\infty} F_n$, it is sufficient to prove that each of the sets F_n is closed in C. Let $n > 1$ be a natural number and let $\overline{F}_n$ be the closure of the set F_n. Let $f \in \overline{F}_n$. Then there is a sequence $\{f_k\}_{k=1,2,\ldots}$ of functions $f_k \in F_n$ such that $\varrho(f_k, f) \to 0$. It is easy to verify that

$$\lim_{k\to\infty} \Phi_{f_k}(x,h) = \Phi_f(x,h)$$

for each $x \in \langle 1/n, 1 - 1/n\rangle$ and each h such that $0 < |h| < 1/n$. The set $\{f_1, f_2, \ldots\}$ is a compact set in C, whence, according to the Arzeli-Ascoli's theorem (see [4], p. 167), for every $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x'| < \delta$ implies $|f_k(x) - f_k(x')| < \varepsilon$ for each $k = 1, 2, \ldots$ Let $x_k \in \langle 1/n, 1 - 1/n\rangle$ be a point with the following property: if $0 < |h| < 1/n$, then $\Phi_{f_k}(x_k, h) \leqslant n$. We may assume that

$$\lim_{k\to\infty} x_k = x_0 \in \langle 1/n, 1 - 1/n\rangle.$$

Let $|x_0 - x_k| < \delta$ for $k \geqslant k_0$. Then

$$|\Phi_{f_k}(x_0, h) - \Phi_{f_k}(x_k, h)| < \varepsilon/|h|$$

holds for $k \geqslant k_0$ and $0 < |h| < 1/n$, whence

$$\Phi_{f_k}(x_0, h) < \Phi_{f_k}(x_k, h) + \varepsilon/|h| \leqslant n + \varepsilon/|h|$$

and

$$\Phi_f(x_0, h) = \lim_{k\to\infty} \Phi_{f_k}(x_0, h) \leqslant n + \varepsilon/|h|.$$

Since the last inequality holds for every $\varepsilon > 0$, $\Phi_f(x_0, h) \leqslant n$. Hence $f \in F_n$.

Hence the set N_1 is of the first category in C. And since N_2 is the isometric image of N_1 in the isometry $T(f) = -f$ of the space C onto itself, also N_2 is of the first category in C.

COROLLARY 1. *The set D_s of all $f \in C$ for which there exists a symmetric derivative in at least one point $x \in (0, 1)$ is a set of the first category in C.*

Proof follows from the inclusion $D_s \subset N$.

COROLLARY 2. *The set D of all $f \in C$ for which there exists a derivative (in the usual sense) in at least one point $x \in (0, 1)$ is a set of the first category in C.*

Proof follows from the inclusion $D \subset D_s$ and corollary 1.

REFERENCES

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