COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 2

# A SURVEY ON AXIOMS OF SUBMANIFOLDS IN RIEMANNIAN AND KAEHLERIAN GEOMETRY

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*DEDICATED TO PROFESSOR KENTARO YANO FOR HIS 70TH BIRTHDAY*

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**1. Introduction.** In his book on Riemannian geometry, E. Cartan singled out the real space forms among all Riemannian manifolds by a characteristic property, called the *axiom of planes*, which basically is a condition to admit sufficiently many totally geodesic submanifolds. Since then, many differential geometers obtained weaker conditions of this nature which are typical of some of the most primitive spaces in their field. In this survey we will discuss as such the real and complex space forms, the conformally flat spaces and the Bochner–Kaehler spaces being determined among all Riemannian or Kaehlerian manifolds by suitable axioms of submanifolds. By a *real space form*, a *complex space form*, a *conformally flat space* and a *Bochner–Kaehler space* we mean a Riemannian manifold of constant sectional curvature, a Kaehlerian manifold of constant holomorphic sectional curvature, a Riemannian manifold which is locally conformal to a Euclidean space and a Kaehlerian manifold with identically vanishing Bochner curvature tensor, respectively.

**2. Some basic facts on Riemannian manifolds.** Let $M$ be a Riemannian manifold of dimension $m$ with metric tensor $g$, corresponding Levi–Cività connection $\nabla$, Riemann–Christoffel curvature tensor $R$, Ricci tensor $S$ and scalar curvature $\varrho$. Then $M$ is called a *space of constant curvature* or a *real space form* if the sectional curvature

$$k(\gamma) = R(X, Y; Y, X) \tag{2.1}$$

for the plane section $\gamma$ determined by orthonormal tangent vectors $X$, $Y$ at a point $p$ of $M$ is a constant for all plane sections $\gamma$ of $T_p M$ and for all points $p$ in $M$. The sectional curvature $k(\gamma)$ is the Gauss curvature at $p$ of the surface consisting of the geodesics of $M$ which pass through $p$ and which are tangent to $\gamma$. By the Lemma of Schur [66], if at each point $p$ of a Riemannian manifold $M$ with dimension $m > 2$ the sectional curvature $k(\gamma)$ depends only on the point $p$, then $M$ is a real space form. Also the following result is well known (see, e.g., [40]).

THEOREM 2.1. *The Riemann–Christoffel curvature tensor of a space of constant curvature c has the typical form*

$$R(A, B; C, D) = c\{g(A, D)g(B, C) - g(A, C)g(B, D)\}, \tag{2.2}$$

*where $A$, $B$, $C$ and $D$ are arbitrary tangent vector fields on $M$.*

The following characterization for the real space forms is due to Cartan [7]:

THEOREM 2.2. *Let $M$ be a Riemannian manifold of dimension $> 2$. Then $M$ is a space of constant curvature if and only if*

$$R(X, Y; Z, X) = 0 \tag{2.3}$$

*for all orthonormal vectors $X$, $Y$ and $Z$ at any point of $M$.*

A space of zero curvature is said to be *locally flat* or *locally Euclidean*.

Let $\sigma$ be a positive function on $M$. Then

$$g^* = \sigma^2 g \tag{2.4}$$

is a new metric tensor on $M$ which assigns to any two vectors $X$ and $Y$ at any point $p$ of $M$ the same angle as $g$ and for which the relation between the lengths of a vector $X$ measured with $g^*$ and $g$ is given by

$$\|X\|^* = \sigma(p)\|X\|. \tag{2.5}$$

Such a transformation of metric is called *conformal*. Putting

$$L(A, B) = -\frac{1}{m-2}S(A, B) + \frac{\varrho}{2(m-1)(m-2)}g(A, B), \tag{2.6}$$

we define the *Weyl conformal curvature tensor* $W$ of $M$ by

$$\begin{aligned} W(A, B; C, D) = R(A, B; C, D) &+ g(B, C)L(A, D) - g(A, C)L(B, D) \\ &+ g(A, D)L(B, C) - g(B, D)L(A, C). \end{aligned} \tag{2.7}$$

The corresponding (1, 3)-tensor is invariant under all conformal transformations of $M$. For $m = 3$, $W$ vanishes identically. For $m = 3$, a meaningful conformal invariant tensor $\bar{W}$ is defined by

$$\bar{W}(A, B, C) = (\nabla_A L)(B, C) - (\nabla_B L)(A, C). \tag{2.8}$$

A Riemannian manifold $M$ is said to be a (locally) *conformally flat space* or a (locally) *conformally Euclidean space* if its metric $g$ is conformally related to a metric $g^*$ which is locally flat. By the existence of isothermal coordinates, every surface is (locally) conformally flat. Regarding the conformal flatness of manifolds of dimension $> 2$ we have the following result of Weyl [82]:

THEOREM 2.3. *A necessary and sufficient condition for a Riemannian manifold of dimension $m > 3$ (respectively, $m = 3$) to be conformally Euclidean is that its Weyl conformal curvature tensor $W$ (respectively, its tensor $\bar{W}$) vanishes identically.*

The following characterization for the conformally Euclidean spaces is due to Schouten [65]:

THEOREM 2.4. *Let $M$ be a Riemannian manifold of dimension $> 3$. Then $M$ is conformally flat if and only if*

$$R(X, Y; Z, U) = 0 \tag{2.9}$$

*for all orthonormal tangent vectors $X$, $Y$, $Z$ and $U$ at any point of $M$.*

Real space forms, the products of real space forms with curves and the products of two real space forms of opposite curvature are examples of conformally Euclidean spaces.

If the Ricci tensor of a Riemannian manifold $M$ is proportional to its metric tensor, say

$$S = \lambda g \tag{2.10}$$

for some function $\lambda$ on $M$, then $M$ is called an *Einstein space*. This condition is automatically satisfied for every surface. For $m \geqslant 3$ the function $\lambda$ is automatically constant. Furthermore, all real space forms are Einsteinian. For manifolds of dimension 3, conversely, every Einstein space is a space of constant curvature. For dimension $\geqslant 3$, the real space forms can be characterized as the conformally flat Einstein space.

A Riemannian manifold $M$ is called *locally symmetric* if its curvature tensor $R$ is a covariant constant ($\nabla R = 0$). A *complete locally symmetric space* is a symmetric space, i.e., a Riemannian manifold $M$ such that for each point $p$ of $M$ there exists an involutive isometry of $M$ having $p$ as an isolated fixed point.

**3. Some basic facts on Kaehlerian manifolds.** Let $\tilde{M}$ be a Kaehlerian manifold of real dimension $2m$ with metric tensor $g$, corresponding Levi-Cività connection $\nabla$, complex structure $J$, Riemann–Christoffel curvature tensor $R$, Ricci tensor $S$ and scalar curvature $\varrho$. The sectional curvature $k(\gamma)$ of $\tilde{M}$ for a plane section $\gamma$ which is holomorphic, that is, which is invariant under the action of $J$, is called a *holomorphic sectional curvature.* A Kaehlerian manifold of constant holomorphic sectional curvature is called a *complex space form.* Every complex space form is an Einstein space. Besides a complex version of the Lemma of F. Schur, also the following result is well known (see, e.g., [40]):

THEOREM 3.1. *The Riemann–Christoffel curvature tensor of a Kaehlerian manifold $M$ of constant holomorphic sectional curvature $c$ has the typical form*

(3.1)
$$R(A, B; C, D) = \frac{c}{4}\{g(A, D)g(B, C) - g(A, C)g(B, D) + g(JA, D)g(JB, C) - g(JA, C)g(JB, D) + 2g(A, JB)g(JC, D)\},$$

*where $A$, $B$, $C$ and $D$ are arbitrary tangent vector fields on $\tilde{M}$.*

THEOREM 3.2. *A Kaehlerian manifold of real dimension $> 4$ is a complex space form if and only if it has a constant antiholomorphic sectional curvature.*

Here by an *antiholomorphic sectional curvature* we mean the sectional curvature $k(\gamma)$ for antiholomorphic (anti-invariant, totally real) plane sections $\gamma$, that is, plane sections $\gamma$ which stand orthogonal to their image under $J$. Let $X$ and $Y$ be orthonormal tangent vectors at some point $p$ of $\tilde{M}$ which span a totally real plane section $\pi$ of $T_p\tilde{M}$ (that is, such that $g(X, X) = g(Y, Y) = 1$ and $g(X, Y) = g(X, JY) = 0$). Then the *totally real bisectional curvature* $B(\pi)$ is defined by

$$B(\pi) = R(X, JX; JY, Y), \tag{3.2}$$

and Houh [36] proved the following

THEOREM 3.3. *A Kaehlerian manifold is a complex space form if and only if it has a constant totally real bisectional curvature.*

The following characterization for the spaces of constant holomorphic sectional curvature was obtained by Ogiue [56] and Nomizu [50].

THEOREM 3.4. *A Kaehlerian manifold $\tilde{M}$ is a complex space form if and only if*

$$R(X, Y; JX, Y) = 0 \tag{3.3}$$

*for all orthonormal vectors $X$, $Y$ which span a totally real plane section of $T_p\tilde{M}$ at an arbitrary point $p$ of $\tilde{M}$.*

Correspondingly to Theorem 2.2 we proved [78] the following

THEOREM 3.5. *A Kaehlerian manifold $\tilde{M}$ of real dimension $> 4$ is a complex space form if and only if*

$$R(X, Y; Z, X) = 0 \tag{3.4}$$

*for all orthonormal vectors* $X$, $Y$ *and* $Z$ *at an arbitrary point* $p$ *of* $\tilde{M}$, *which span a totally real subspace of* $T_p\tilde{M}$.

The complex analogue for Kaehlerian manifolds of Weyl's tensor for Riemannian manifolds is given by the *Bochner curvature tensor* $B$ defined by

$$
\begin{aligned}
(3.5)\quad & B(P, Q; C, D) \\
&= R(P, Q; C, D) - \frac{1}{2(m+2)}\{g(Q, C)S(P, D) - g(Q, D)S(P, C) \\
&\quad + g(P, D)S(Q, C) - g(P, C)S(Q, D) + g(JQ, C)S(JP, D) \\
&\quad - g(JQ, D)S(JP, C) + g(JP, D)S(JQ, C) - g(JP, C)S(JQ, D) \\
&\quad - 2g(JC, D)S(JP, Q) - 2g(JP, Q)S(JC, D)\} \\
&\quad + \frac{\varrho}{4(m+1)(m+2)}\{g(Q, C)g(P, D) - g(P, C)g(Q, D) \\
&\quad + g(JQ, C)g(JP, D) - g(JP, C)g(JQ, D) \\
&\quad + 2g(P, JQ)g(JC, D)\},
\end{aligned}
$$

where $P$, $Q$, $C$ and $D$ are tangent vector fields on the Kaehlerian manifold (see, e.g., [5], [86], [85], [70]). A Kaehlerian manifold with identically vanishing Bochner tensor is called a *Bochner–Kaehler manifold* and also is said to be *Bochner flat*. Correspondingly to Theorem 2.4, Yano and Sawaki [91] proved the following

THEOREM 3.6. *A Kaehlerian manifold* $\tilde{M}$ *of real dimension* $> 6$ *is Bochner flat if and only if*

$$(3.6)\qquad R(X, Y; Z, U) = 0$$

*for all orthonormal vectors* $X$, $Y$, $Z$ *and* $U$ *at an arbitrary point* $p$ *of* $\tilde{M}$, *which span a totally real subspace of* $T_p\tilde{M}$.

All complex space forms and all the products of two complex forms of opposite curvature are Bochner–Kaehler manifolds, as a matter of fact these are the only Bochner flat spaces with constant scalar curvature. The complex space forms can be characterized as the Bochner–Kaehler Einstein space.

**4. Some fundamental types of submanifolds.** Let $N$ be an $n$-dimensional submanifold of an $m$-dimensional Riemannian manifold $M$. Then $q = m-n$ is the codimension of $N$ in $M$. Let $g'$ denote the metric tensor induced on $N$ from $g$. By $\nabla'$, $R'$, etc., we denote the corresponding Levi-Cività connection, Riemann–Christoffel curvature tensor, etc., of $N$.

For vector fields $X$ and $Y$ tangent to $N$, the *formula of Gauss* gives the decomposition of the vector field $\nabla_X Y$, which is tangent to $M$, in its components which are tangent and normal, respectively, to the submanifold:

$$(4.1)\qquad \nabla_X Y = \nabla'_X Y + h(X, Y).$$

The normal bundle valued symmetric 2-form $h$ on $N$ is called the *second fundamental form* of $N$ in $M$. For vector fields $X$ and $\eta$ which are tangent and normal, respectively, to $N$, the *formula of Weingarten* gives the decomposition of the vector field $\nabla_X \eta$, which is tangent to $M$, in its components which are tangent and normal, respectively, to the submanifold:

$$\nabla_X \eta = -A_\eta X + \nabla^\perp_X \eta. \tag{4.2}$$

The symmetric linear transformation $A_\eta$ of the tangent space $T_p N$ of $N$ at each of its point $p$ is called the *second fundamental tensor* of $N$ with respect to $\eta$. One has the following relation between the second fundamental form $h$ and the second fundamental tensors $A$:

$$h(X, Y) = \sum_{t=1}^{q} g(A_{\xi_t} X, Y)\xi_t, \tag{4.3}$$

where $\xi_1, \xi_2, \ldots, \xi_q$ is a local field of orthonormal frames of the normal bundle $T^\perp N$ of $N$ in $M$. Let $\xi$ be a unit normal vector of $N$ at some point $p$. Then, since $A_\xi$ is self-adjoint, there exist orthonormal tangent vectors $E_1, E_2, \ldots, E_n$ of $N$ at $p$ which are eigenvectors of $A_\xi$, i.e., such that

$$A_\xi E_i = \lambda_i E_i \tag{4.4}$$

for real numbers $\lambda_i$, $i \in \{1, 2, \ldots, n\}$. $\lambda_i$ and $E_i$ are called the *principal curvatures* and the *principal directions*, respectively, of the normal vector $\xi$ of $N$ at $p$. For a surface $N$ in $E^3$, $\lambda_1$ and $\lambda_2$ are the extremal values of the curvatures of the normal sections of $N$ at $p$, that is, of the curves in which the planes through $\xi$ intersect $N$. The connection $\nabla^\perp$ is called the *normal connection* of $N$ in $M$. This is a metric connection in $T^\perp N$ with respect to the metric induced from $g$. If at each point $p$ in $N$ the first normal space of $N$ in $M$ coincides with the normal space of $N$ in $M$, then, similarly to the uniqueness theorem for the Riemannian connection, the normal connection $\nabla^\perp$ can be characterized as the unique metric linear connection in $T^\perp N$ whose torsion tensor is zero [52]. The corresponding curvature tensor $R^\perp$,

$$R^\perp(X, Y)\eta = \nabla^\perp_X \nabla^\perp_Y \eta - \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_{[X,Y]} \eta, \tag{4.5}$$

is called the *normal curvature tensor* of $N$ in $M$. The normal connection is said to be *flat* or *trivial* if $R^\perp$ vanishes identically. A unit normal vector field $\xi$ on $N$ is said to be *parallel* (in the normal bundle) if

$$\nabla^\perp_X \xi = 0 \tag{4.6}$$

for all $X$ tangent to $N$. For submanifolds $N$ of an arbitrary Riemannian manifold $M$, the normal connection is flat if and only if there exist locally $q$ mutually orthogonal parallel unit normal vector fields on $N$. For submanifolds of a conformally flat space, the flatness of normal connection is equivalent to the simultaneous diagonalizability of all second fundamental tensors [10]. The second fundamental form $h$ and the normal connection $\nabla^{\perp}$ describe the extrinsic properties of $N$ in $M$.

Next we state the fundamental equations for a submanifold $N$ of a Riemannian manifold $M$. These are the *equations of Gauss*:

$$R(X, Y; Z, W) = R'(X, Y; Z, W) + g\big(h(X, Z), h(Y, W)\big) - g\big(h(X, W), h(Y, Z)\big), \tag{4.7}$$

the *equations of Codazzi*:

$$R(X, Y; Z, \eta) = g\big((\bar{\nabla}_X h)(Y, Z), \eta\big) - g\big((\bar{\nabla}_Y h)(X, Z), \eta\big), \tag{4.8}$$

and the *equations of Ricci*:

$$R(X, Y; \eta, \zeta) = R^{\perp}(X, Y; \eta, \zeta) - g'([A_\eta, A_\zeta] X, Y), \tag{4.9}$$

where $X, Y, Z, W$ and $\eta, \zeta$ are vector fields tangent and normal to $N$, respectively, $\bar{\nabla}_X h$ is the covariant derivative of van der Waerden–Bortolotti of $h$ with respect to $X$, that is

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^{\perp}\big(h(Y, Z)\big) - h(\nabla'_X Y, Z) - h(Y, \nabla'_X Z), \tag{4.10}$$

and $[A_\eta, A_\zeta]$ is the Lie bracket of $A_\eta$ and $A_\zeta$ ([9], [15]).

For $n$-dimensional submanifolds $N$ of a (real) $2m$-dimensional Kaehlerian manifold $\tilde{M}$ distinctions are made according to the behaviour of the tangent bundle $TN$ of $N$ with respect to the complex structure $J$ of $\tilde{M}$. If $TN$ is invariant under the action of $J$ ($J(TN) = TN$), then $N$ is called a *Kaehlerian* (*holomorphic, complex, invariant*) *submanifold* of $M$ (see [57]). Endowed with the restriction of $J$ to $TN$, in this case $N$ also becomes a Kaehlerian manifold, and so the dimension of $N$ is necessarily even. If $J$ transforms $TN$ into the normal bundle $T^{\perp}N$ ($J(TN) \subset T^{\perp}N$), then $N$ is called a *totally real* (*antiholomorphic, anti-invariant*) *submanifold* of $M$. An immediate consequence of this definition is that necessarily $1 \leqslant n \leqslant m$. For a normal vector field $\xi$ on a totally real submanifold $N$ of $\tilde{M}$, $J\xi$ can be decomposed as

$$J\xi = P\xi + f\xi, \tag{4.11}$$

where $P\xi$ and $f\xi$ denote the tangential and normal components of $J\xi$, respectively. $f$ determines an *f-structure* in $T^{\perp}N$, namely an endomorphism of the normal bundle which satisfies $f^3 + f = 0$. If

$$(D_X f)\xi := \nabla_X^{\perp} f\xi - f\nabla_X^{\perp}\xi = 0 \tag{4.12}$$

for all tangent vector fields $X$ on $N$, then the structure $f$ is said to be *parallel* [87]. The submanifolds $N$ of $\tilde{M}$ for which $TN$ can be decomposed into the direct sum of a holomorphic distribution $\mathscr{D}$ and orthogonal antiholomorphic distribution $\mathscr{D}^{\perp}$ are called *CR-submanifolds*. If $\mathscr{D} \neq \{0\}$ and

$\mathscr{D}^\perp \neq \{0\}$, then $N$ is called a *proper CR-submanifold*. CR-submanifolds are CR-manifolds in the sense of [32]. The real hypersurfaces of a Kaehler space constitute examples of proper CR-submanifolds [88].

After this short survey of formulas and definitions for submanifolds of Riemannian and Kaehlerian spaces, we now briefly discuss different types of submanifolds which appear in the axioms of submanifolds in Riemannian and Kaehlerian geometry. In some sense, the most elementary submanifolds $N$ of a Riemannian manifold $M$ are the *totally geodesic* ones. They can be thought of as the submanifolds $N$ of $M$ such that in order to travel from one point in $N$ to another point in $N$ by the shortest way in $M$ one must not leave $N$. More precisely, they are the submanifolds $N$ of $M$ whose geodesics are also geodesics in $M$. By the formula of Gauss one may see that the totally geodesic submanifolds $N$ of $M$ are characterized by the vanishing of their second fundamental form $h$. The totally geodesic submanifolds of Euclidean spaces $E^m$, spheres $S^m$ and real projective spaces $\boldsymbol{RP}^m$ were classified by Cartan [7]. Wolf [83] classified the totally geodesic submanifolds of the complex projective spaces $\boldsymbol{CP}^m$, the quaternionic projective spaces $\boldsymbol{HP}^m$ and the Cayley plane $OP^2$. Together these results give complete information on the totally geodesic submanifolds of the Euclidean spaces and the symmetric spaces of rank 1. The *rank of a symmetric space* $M$ is the maximal dimension of flat totally geodesic submanifolds of $M$. Totally geodesic submanifolds of symmetric spaces of arbitrary rank were studied by Chen and Nagano [16] by their $(M_+, M_-)$-theory. A normal direction $\xi$ on a submanifold $N$ in a Riemannian manifold $M$ is said to be *geodesic* if

$$A_\xi = 0. \tag{4.13}$$

Thus $N$ is totally geodesic in $M$ if and only if all normal sections $\xi$ are geodesic. By Yano and Kon [87], Hendrickx and one of the authors [34], the parallelism of the $f$-structure in the normal bundle of a totally real submanifold $N$ of a Kaehlerian manifold $\tilde{M}$ is equivalent to the property that $N$ is geodesic with respect to all normal sections of $T^\perp N \backslash J(TN)$.

After the totally geodesic submanifolds, in some sense, the most simple submanifolds are the *totally umbilical* ones. A recent survey on totally umbilical submanifolds was given by Chen [14]. Consider a hypersphere $S^n(r)$ of radius $r$ which is centered at the origin of $E^{n+1}$. Let $\xi$ be the inner unit normal vector field of $S^n(r)$:

$$\xi = -\frac{1}{r}\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}, \tag{4.14}$$

where $\partial/\partial x_i$ is the natural frame field of $E^{n+1}$. The covariant derivative $\nabla_X \xi$ of $\xi$ with respect to a vector

$$X = \sum_{i=1}^{n+1} X_i \frac{\partial}{\partial x_i} \tag{4.15}$$

tangent to $S^n(r)$ is given by

$$(4.16)\qquad \nabla_X\xi = \left(\sum_{i=1}^{n+1} X_i\frac{\partial}{\partial x_i}\right)\left(-\frac{1}{r}\sum_{j=1}^{n+1} x_j\frac{\partial}{\partial x_j}\right)$$

$$= -\frac{1}{r}\sum_{i,j=1}^{n+1} X_i\frac{\partial x_j}{\partial x_i}\frac{\partial}{\partial x_j} = -\frac{1}{r}\sum_{i,j=1}^{n+1} X_i\delta_{ij}\frac{\partial}{\partial x_j}$$

$$= -\frac{1}{r}\sum_{j=1}^{n+1} X_j\frac{\partial}{\partial x_j} = -\frac{1}{r}X.$$

Of course, since $S^n(r)$ is of codimension 1 in $E^{n+1}$ and $\xi$ is of unit length, $\nabla_X\xi$ has no component normal to $S^n(r)$. From (4.16) and the formula of Weingarten we have

$$(4.17)\qquad A_\xi X = \frac{1}{r}X$$

for all vectors tangent to $S^n(r)$. Thus

$$(4.18)\qquad A_\xi = \frac{1}{r}\mathrm{Id},$$

where Id denotes the identity transformation. Consequently,

$$(4.19)\qquad h(X, Y) = \frac{1}{r}g'(X, Y)\xi.$$

In general, for any submanifold $N$ of a Riemannian manifold $M$,

$$(4.20)\qquad H = \frac{1}{n}\operatorname{trace} h$$

is a canonically determined normal vector field on $N$ in $M$. It is called the *mean curvature vector field* of $N$. With respect to an orthonormal frame field $\xi_t$ it is given by

$$(4.21)\qquad H = \frac{1}{n}\sum_{t=1}^{q}(\operatorname{trace} A_{\xi_t})\,\xi_t.$$

A submanifold $N$ of $M$ is said to be *minimal* if $H$ vanishes identically. Every complex submanifold of a Kaehlerian manifold is minimal. In the present situation, for $S^n(r)$ in $E^{n+1}$, we see from (4.18) that

$$(4.22)\qquad \operatorname{trace} A_\xi = \frac{n}{r},$$

so that

$$(4.23)\qquad H = \frac{1}{r}\xi.$$

Combining this with (4.19) we obtain

$$h(X, Y) = g'(X, Y)H. \tag{4.24}$$

In general, a submanifold $N$ of a Riemannian manifold $M$ is called *totally umbilical* if it satisfies (4.24). Clearly, the totally umbilical minimal submanifolds $N$ of $M$ are the totally geodesic submanifolds of $M$. From (4.24) we see that for each unit normal $\xi$ on a totally umbilical submanifold the second fundamental tensor $A_\xi$ is proportional to the identity transformation. In other words, for a totally umbilical submanifold every $A_\xi$ has an eigenvalue with multiplicity $n$; in particular, for a totally geodesic $N$ every $A_\xi$ has 0 as the only eigenvalue. Since the mean curvature $\|H\| = 1/r$ is constant and the codimension is 1, it is clear from (4.23) that for any sphere $S^n(r)$ in $E^{n+1}$ the normal component of $\nabla_X H$ vanishes for all $X$ tangent to $S^n(r)$, i.e., that

$$\nabla_X^\perp H = 0. \tag{4.25}$$

In general, the totally umbilical submanifolds $N$ of a Riemannian manifold $M$ with parallel non-vanishing mean curvature vector $H$ are called *extrinsic spheres*. For arbitrary Riemannian manifolds the extrinsic spheres are the natural analogues of the ordinary spheres in Euclidean spaces ([51], [54]). The totally umbilical submanifolds of Euclidean spaces are classified as follows [7]: an $n$-dimensional submanifold $N$ of a Euclidean space $E^m$ is totally umbilical if and only if $N$ is either an $n$-plane or an ordinary $n$-sphere. The classification of totally umbilical submanifolds in other real space forms is very similar to the one in $E^m$ (see [9]). In particular, totally umbilical submanifolds of real space forms are themselves also real space forms. Totally umbilical submanifolds in the other symmetric spaces of rank 1 are classified in [18], [13] and [12]. For instance, Chen and Ogiue [18] proved the following:

A totally umbilical submanifold $N$ of dimension $n > 2$ in a complex projective space $\boldsymbol{C}P^m$ is one of the following:

(i) a complex projective space $\boldsymbol{C}P^{n/2}$ holomorphically immersed in $\boldsymbol{C}P^m$ as a totally geodesic submanifold,

(ii) a real projective space $\boldsymbol{R}P^n$ immersed in $\boldsymbol{C}P^m$ as a totally real and totally geodesic submanifold,

(iii) a real projective space $\boldsymbol{R}P^n$ immersed in $\boldsymbol{C}P^m$ as a totally real extrinsic sphere.

In particular, totally umbilical submanifolds of complex space forms are real space forms or complex space forms. By combining results of Chen and Nagano [16] and Verheyen and one of the authors [76], the (real) dimension of an extrinsic sphere in a positively (or negatively) curved Kaehlerian manifold $\tilde{M}$ must be smaller than the complex dimension of $\tilde{M}$. In this context, Blair and Chen [4] showed that there exist no totally umbilical proper CR-submanifolds in any positively (or negatively) curved Kaehlerian manifold. A Riemannian manifold is called an *intrinsic sphere* if it is locally isometric to a standard sphere in a Euclidean space. It seems natural to ask when an extrinsic sphere is an intrinsic sphere. In this respect we mention the following result of Chen [11]. Let $N$ be a complete simply-connected even-dimensional submanifold with flat normal connection in any Kaehler manifold. Then, if $N$ is an extrinsic sphere, then $N$ is an intrinsic sphere.

A submanifold $N$ of a Riemannian manifold $M$ has the *parallel second fundamental form* if

$$\bar{\nabla} h = 0. \tag{4.26}$$

For a totally umbilical submanifold $N$, (4.26) holds if and only if $N$ is either totally geodesic or if $N$ is an extrinsic sphere [31]. The submanifolds with parallel second fundamental form can be characterized as the *extrinsic locally symmetric* submanifolds, that is, roughly speaking, as the submanifolds which are locally invariant under the reflections into their normal spaces at each of their points ([29], [69], [68]). Of course, extrinsic symmetric submanifolds also are (intrinsic) symmetric spaces. The classification of these submanifolds in real space forms was given by Ferus [26]–[28] and by Backes and Reckziegel [1], and Takeuchi [72], Nakagawa and Takagi [49], and Kon [41]. For studies of (nonzero) isotropic submanifolds with parallel second fundamental form in symmetric spaces, see [47], [48], [53] and [37]. The latter submanifolds have the property that every geodesic in the submanifold is a circle in the ambient space [53], and examples in real space forms are given by the submanifolds with planar geodesics (see [59], [62], [35]). Here by a *($\lambda$-)isotropic submanifold* $N$ of a Riemannian manifold is meant a submanifold such that for a given $\lambda \geqslant 0$, for all unit tangent vectors $X$ at all points $p$ of $N$ we have

$$\|h(X, X)\| = \lambda, \tag{4.27}$$

where $\|\xi\| = g(\xi, \xi)^{1/2}$ (see [58]).

A submanifold $N$ of a Riemannian manifold $M$ satisfies the *classical Codazzi equation* if for all tangent vector fields $X$, $Y$ and $Z$ on $N$ we have

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{4.28}$$

In particular, this is evidently the case whenever $h$ is parallel. Examples of submanifolds satisfying the classical Codazzi equation are given by all Kaehler submanifolds in any complex space form [69].

After the umbilical submanifolds, in some sense, the most simple submanifolds are the *quasiumbilical* ones. According to Chen and Yano [23], [24], a hypersurface $N$ of a Riemannian manifold $M$ is said to be *quasiumbilical* if it has a principal curvature with multiplicity $\geqslant n-1$, i.e., if the principal curvatures of $N$ are given by $\mu, \lambda, \ldots, \lambda$, where $\lambda$ occurs $n-1$ times (see also [20]). Obviously, this definition is meaningful only for hypersurfaces with dimension $\geqslant 3$. It is clear that quasiumbilicity may be considered as a generalization of umbilicity, in which case $\mu = \lambda$. On the other hand, the notion of quasiumbilicity may be considered as a generalization of the notion of *cylindricity*, the principal curvatures of a cylindrical hypersurface $C$ of a Euclidean space $E^{n+1}$ being given by $\mu, 0, \ldots, 0$, where 0 occurs $n-1$ times. A cylindrical hypersurface or, shortly, a hypercylinder $C$ of $E^{n+1}$ is obtained by moving parallelly an $(n-1)$-dimensional linear subspace of $E^{n+1}$ along some curve $\varkappa$. In particular, if $\varkappa$ is a straight line, then $C$ is a hyperplane in $E^{n+1}$, i.e., then the hypersurface is totally geodesic ($\mu = 0$). From the equation of Gauss one may see that a hypersurface $N$ in a Riemannian manifold $M$ is a hypercylinder if and only if the curvature tensors $R'$ and $R$ are equal on $TN$, that is, if $R'(X, Y; Z, W) = R(X, Y; Z, W)$ for all vectors $X, Y, Z, W$ tangent for $N$ (see [22]). From the contraction of the equation of Gauss (see, e.g., [55]) one may see that, for $n \geqslant 3$, a hypersurface $N$ of a Riemannian manifold $M$ is cylindrical if and only if $\mathrm{III} = \mathrm{II}_H$, where III and $\mathrm{II}_H$ are the third fundamental form and the quadratic mean form of $N$ in $M$, respectively [22]. In particular, for $n \geqslant 3$, a hypersurface $N$ of $E^{n+1}$ is cylindrical if and only if $N$ is Ricci flat [40]. A geometrical motivation for considering quasiumbilicity as a generalization of both umbilicity and cylindricity is given by the following result of Chen and Yano [23], [9], and Kulkarni [43]: every quasiumbilical hypersurface of a real space form $M$ admits a codimension 1-foliation the leaves of which are totally umbilical submanifolds of $M$. Well-known examples of quasiumbilical hypersurfaces of the Euclidean space $E^{n+1}$ are the canal hypersurfaces [25], i.e., the envelopes of 1-parameter families of hyperspheres in $E^{n+1}$. In this respect we also mention that Blair [3] showed that the minimal quasiumbilical hypersurfaces of Euclidean spaces are either totally geodesic or generalized catenoids (see also [59]). Further examples of quasiumbilical hypersurfaces are given by all geodesic hyperspheres in any complex space form [71]. A submanifold $N$ of a Riemannian manifold $M$ with arbitrary codimension $q$ is said to be *quasiumbilical with respect to a normal section* $\xi$, and $\xi$ is called a *quasiumbilical normal section* of $N$, if the principal curvatures of $N$ with respect to $\xi$ are given by $\mu_\xi, \lambda_\xi, \ldots, \lambda_\xi$, where $\lambda_\xi$ occurs $n-1$ times. In particular, $\xi$ is said to be a *cylindrical, umbilical* or *geodesic normal section* if $\lambda_\xi = 0$, $\mu_\xi = \lambda_\xi$ or $\mu_\xi = \lambda_\xi = 0$, respectively. $N$ is said to be a *totally quasiumbilical* (*cylindrical*) *submanifold* of $M$ if there exist $q$ mutually orthogonal quasiumbilical (cylindrical) normal sections on $N$. A quasiumbilical normal section $\xi$ of $N$ which is not umbilical is said to be *properly quasiumbilical.* The principal direction $Z$ of $N$ with respect to a proper quasiumbilical normal section $\xi$ and corresponding to the principal curvature $\mu_\xi$ is called the *distinguished direction of N with respect to* $\xi$, and in this situation $N$ is said to be *Z-quasiumbilical with respect to* $\xi$. As asserted in the following results, the extrinsic property of quasiumbilicity is closely related to the intrinsic property of conformal flatness. When $n > 3$, every conformally flat hypersurface of a conformally flat space is quasiumbilical ([6], [64]), and every totally quasiumbilical submanifold of a conformally flat space is conformally flat [24]. In a 4-dimensional Euclidean space, however, there do exist conformally Euclidean hypersurfaces which are not quasiumbilical [44]. With respect to the equivalence of conformal flatness and quasiumbilicity for submanifolds of dimension $n > 3$, one also has the following results. Every conformally flat submanifold with codimension $q \leqslant \min\{4, n-3\}$ in a conformally flat space is totally quasiumbilical [46], and every conformally flat submanifold with codimension $q \leqslant n-3$ and with flat normal connection in a conformally flat space is totally quasiumbilical [21]. We remark that the properties of umbilicity, quasiumbilicity, flatness of normal connection and commutativity of second fundamental tensors (in contrast with geodesicness, cylindricity and, for example, also minimality) are all invariant under the conformal changes of the metric on the ambient space [10]. Finally, we mention that, in a Bochner–Kaehler space, every totally quasiumbilical totally real submanifold of dimension $> 3$ is conformally flat [81].

**5. Axioms of submanifolds in Riemannian geometry.** According to E. Cartan, a Riemannian manifold $M$ satisfies the *axiom of n-planes* if for each point $p$ in $M$ and for every $n$-dimensional linear subspace $T$ of $T_p M$ there exists an $n$-dimensional totally geodesic submanifold $N$ of $M$ passing through $p$ and such that $T_p N = T$ (where $n$ is a fixed integer). Moreover, he proved the following

THEOREM 5.1 (Cartan [7]). *A Riemannian manifold of dimension $m \geqslant 3$ satisfies the axiom of n-planes for some $n$, $2 \leqslant n < m$, if and only if it is a real space form.*

The axiom of planes was originally introduced by Riemann [61] in postulating the existence of a surface $S$ passing through three given points with the property that every straight line having two points in $S$ is completely contained in this surface. Beltrami [2] proved that a space of constant curvature satisfies the axiom of 2-planes, and Schur [66] proved the converse. Actually the latter result was also obtained by Schlaefli [63] in combination with work of Klein [39]. Hereby the rôle of straight lines is played by geodesics. In this respect, see also [8].

As a generalization of the axiom of $n$-planes, D. S. Leung and K. Nomizu in 1971 introduced the *axiom of n-spheres*: for each point $p$ in $M$ and for every $n$-dimensional linear subspace $T$ of $T_p M$ there exists an $n$-dimensional totally umbilical submanifold $N$ of $M$ with parallel mean curvature vector field such that $p \in N$ and $T_p N = T$, and they proved the following

THEOREM 5.2 (Leung and Nomizu [45]). *A Riemannian manifold of dimension $m \geqslant 3$ satisfies the axiom of n-spheres, $2 \leqslant n < m$, if and only if it is a real space form.*

Since the mean curvature vector field of a hypersurface is parallel if and only if the mean curvature is constant, in case $n = m-1$ Theorem 5.2 was originally due to Schouten [65]; we remark that this particular case was also treated by Kowalski [42].

Further generalizations in this direction were given as follows by S. I. Goldberg and E. M. Moskal in 1976 and by W. Strübing in 1979.

THEOREM 5.3 (Goldberg and Moskal [31]). *A Riemannian manifold M of dimension $m \geqslant 3$ is a real space form if and only if for every point $p \in M$ and for each n-dimensional linear subspace T of $T_p M$, $2 \leqslant n < m$, there exists an n-dimensional submanifold N with parallel second fundamental form in M such that $p \in N$ and $T_p N = T$.*

THEOREM 5.4 (Strübing [69]). *A Riemannian manifold M of dimension $m \geqslant 3$ is a real space form if and only if for every point $p \in M$ and for each n-dimensional linear subspace T of $T_p M$, $2 \leqslant n < m$, there exists an n-dimensional submanifold N in M which satisfies the classical Codazzi equation and such that $p \in N$ and $T_p N = T$.*

Indeed, a totally umbilical submanifold has a parallel mean curvature vector field if and only if the second fundamental form is parallel, and in this case the classical Codazzi equation is trivially satisfied.

With respect to the axiom of spheres, the weaker axioms, obtained by dropping the condition that the submanifolds $N$ should have parallel mean curvature vector field or by replacing this condition by the one according to which for any vector $V$ perpendicular to $T$ at $p$ there exists a totally umbilical submanifold $N$ with $p \in N$ and $T_p N = T$ such that $V$ is the mean curvature vector of $N$ at $p$, give the following characterizations for the conformally flat spaces established, respectively, by J. A. Schouten in 1924, K. L. Stellmacher in 1951 and K. Yano and Y. Mutô in 1941.

THEOREM 5.5 (Schouten [64]). *A Riemannian manifold of dimension $m \geqslant 4$ is conformally flat if and only if it satisfies the axiom of totally umbilical submanifolds of dimension n, $3 \leqslant n < m$.*

THEOREM 5.6 (Stellmacher [67]). *A 3-dimensional Riemannian manifold is conformally flat if and only if it satisfies the axiom of totally umbilical surfaces.*

THEOREM 5.7 (Yano and Mutô [90]). *A Riemannian manifold of dimension $\geqslant 4$ is conformally flat if and only if it satisfies the axiom of totally umbilical surfaces with prescribed mean curvature vector.*

Based on the conformal invariance of the notion of quasiumbilicity, it can be observed that for every point $p$ in any conformally flat space $M$ with dimension $> 3$ and for every $(m-1)$-dimensional linear subspace $T$ of $T_p M$ there exist quasiumbilical hypersurfaces $N$ in $M$ such that $p \in N$ and $T_p N$

$= T$. Since every geodesic hypersphere in a complex space form is $J\xi$-quasiumbilical (where $\xi$ is the hypersurface normal), this property also holds for the nonflat complex space forms. This implies that for $n = m-1$ Theorem 5.5 can only partially be generalized from umbilical hypersurfaces to quasiumbilical ones, and that in order to obtain a property which is characteristic of conformally flat spaces it is necessary to impose an additional condition on the quasiumbilical hypersurfaces. We recall that a hypersurface $N$ of a conformally flat space $M$ of dimension $m > 4$ is quasiumbilical if and only if $N$ is conformally flat, which suggests to use conformal flatness as this additional condition. In doing so, in 1981 we proved the following result:

THEOREM 5.8 (Van Lindt and Verstraelen [77]). *A Riemannian manifold of dimension $m > 4$ is conformally flat if and only if it satisfies the axiom of conformally flat totally quasiumbilical submanifolds of dimension $n$, $3 < n < m$.*

Hereby a Riemannian manifold $M$ is said to satisfy the *axiom of conformally flat totally quasiumbilical submanifolds of dimension* $n \geqslant 3$ if for every point $p \in M$ and for every $n$-dimensional linear subspace $T$ of $T_p M$ there exists a conformally flat totally quasiumbilical submanifold $N$ of dimension $n$ such that $p \in N$ and $T_p N = T$.

By the intrinsic characterization in terms of $R$ and $R'$ of the hypercylinders of a Riemannian manifold it is clear that hypercylinders in conformally flat spaces and real space forms are themselves conformally flat spaces and real space forms, respectively. In some sense, conversely, this result also shows that the conformally flat spaces and the real space forms can be characterized by an axiom of conformally flat hypercylinders and an axiom of hypercylinders with constant sectional curvature, respectively. Theorem 5.8 gives an improvement of the first of these statements, and our next result of 1981 does so for the second one. For its formulation we give the following definition. A Riemannian manifold $M$ satisfies the *axiom of Einsteinian totally cylindrical submanifolds of dimension* $n \geqslant 3$ if for every point $p \in M$ and for every $n$-dimensional linear subspace $T$ of $T_p M$ there exists an Einsteinian totally cylindrical submanifold $N$ of dimension $n$ such that $p \in N$ and $T_p N = T$.

THEOREM 5.9 (Van Lindt and Verstraelen [77]). *A Riemannian manifold of dimension $m > 3$ is a real space form if and only if it satisfies the axiom of Einsteinian totally cylindrical submanifolds of dimension $n$, $2 < n < m$.*

Finally, we state the following characterization for the conformally flat spaces which was given in 1975 by B. Y. Chen and one of the authors.

THEOREM 5.10 (Chen and Verstraelen [19]). *A Riemannian manifold $M$ of dimension $m > 3$ is conformally flat if and only if for every point $p \in M$ and every $n$-dimensional linear subspace $T$ of $T_p M$ there exists an $n$-dimensional submanifold $N$ of $M$, $2 \leqslant n < m$, which passes through $p$ and which at $p$ is tangent to $T$ such that $N$ has flat normal connection and commutative second fundamental tensors.*

In relation with Theorems 5.5 and 5.7 we recall that every totally umbilical submanifold of a conformally flat space has flat normal connection and commutative second fundamental tensors.

**6. Axioms of submanifolds in Kaehlerian geometry.** Many authors in various ways adapted the axioms of submanifolds in Riemannian geometry to Kaehlerian geometry.

In 1955, K. Yano and I. Mogi defined a Kaehlerian manifold $\tilde{M}$ to satisfy the *axiom of holomorphic 2n-planes* if through each point $p \in \tilde{M}$ and a tangent vector to any holomorphic linear subspace $T$ of $T_p\tilde{M}$ with dimension $2n$ there passes a totally geodesic submanifold of $\tilde{M}$ having $T$ as a tangent space, and they characterized the complex space forms as follows:

THEOREM 6.1 (Yano and Mogi [89]). *A Kaehlerian manifold of real dimension $2m \geqslant 4$ satisfies the axiom of holomorphic 2n-planes for some n, $1 \leqslant n < m$, if and only if it is a complex space form.*

Independently, K. Nomizu on the one hand and B. Y. Chen and K. Ogiue on the other hand in 1973 introduced the *axiom of antiholomorphic n-planes* by requiring that the linear subspaces $T$ under consideration are totally real.

THEOREM 6.2 (Chen and Ogiue [17], Nomizu [50]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $2m \geqslant 4$ is a complex space form if and only if for every point $p \in \tilde{M}$ and for each n-dimensional antiholomorphic linear subspace $T$ of $T_p\tilde{M}$, $2 \leqslant n \leqslant m$, there exists an n-dimensional totally geodesic submanifold $N$ in $\tilde{M}$ such that $p \in N$ and $T_pN = T$.*

In 1973, respectively in 1976, S. I. Goldberg, respectively S. I. Goldberg and E. M. Moskal gave the following definition of the *axiom of holomorphic 2n-spheres*: for each point $p \in \tilde{M}$ and for every $2n$-dimensional holomorphic linear subspace $T$ of $T_p\tilde{M}$, there exists a $2n$-dimensional totally umbilical submanifold $N$ of $M$ with parallel mean curvature vector field such that $p \in N$ and $T_pN = T$, and they proved the following

THEOREM 6.3 (Goldberg [30], Goldberg and Moskal [31]). *A Kaehlerian manifold of real dimension $2m \geqslant 4$ satisfies the axiom of holomorphic 2n-spheres, $1 \leqslant n < m$, if and only if it is a complex space form.*

For the antiholomorphic case, M. Harada in 1974, S. Yamaguchi and M. Kon in 1978, and the authors in 1981 obtained the following results:

THEOREM 6.4 (Harada [33]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $2m \geqslant 4$ is a complex space form if and only if for every point $p \in \tilde{M}$ and for each n-dimensional antiholomorphic linear subspace $T$ of $T_p\tilde{M}$, $2 \leqslant n \leqslant m$, there exists a totally umbilical submanifold $N$ of $\tilde{M}$ with parallel mean curvature vector field such that $p \in N$ and $T_pN = T$.*

THEOREM 6.5 (Yamaguchi and Kon [84]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $2m \geqslant 4$ is a complex space form if and only if for every point*

*$p \in \tilde{M}$ and for each n-dimensional antiholomorphic linear subspace $T$ of $T_p\tilde{M}$, $2 \leqslant n \leqslant m$, there exists a totally umbilical anti-invariant submanifold $N$ of $\tilde{M}$ with $p \in N$ and $T_pN = T$.*

THEOREM 6.6 (Van Lindt and Verstraelen [78]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $2m > 4$ is a complex space form if and only if for every point $p \in \tilde{M}$ and for each n-dimensional antiholomorphic linear subspace $T$ of $T_p\tilde{M}$, $2 \leqslant n \leqslant m$, there exists a totally real submanifold $N$ of $\tilde{M}$ with commutative second fundamental tensors and parallel f-structure in the normal bundle such that $p \in N$ and $T_pN = T$.*

For dimensions $m \geqslant 3$ and $2 \leqslant n < m$, in 1982, O. Kassabov obtained a characterization for the complex space forms by dropping the condition of the parallelism of the mean curvature vector field $H$ in the axiom of holomorphic $2n$-spheres. We remark however that, on the remaining hypothesis and for the given dimensions, automatically $\nabla_X^\perp H = 0$ for all $X \in T_pN$, and a similar observation can be made in the antiholomorphic case [79].

THEOREM 6.7 (Kassabov [38]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $2m > 4$ is a complex space form if and only if for every 2n-dimensional holomorphic (respectively, n-dimensional antiholomorphic) linear subspace $T$ of $T_p\tilde{M}$ at any point $p \in \tilde{M}$, $2 \leqslant n < m$, there exists a 2n-dimensional (respectively, n-dimensional) totally umbilical submanifold $N$ such that $p \in N$ and $T_pN = T$.*

Corresponding to Theorem 5.4, in 1983, P. Verheyen and one of the authors proved the following

THEOREM 6.8 (Verheyen and Verstraelen [79]). *Let $\tilde{M}$ be a Kaehlerian manifold of dimension $2m \geqslant 4$ and let n be a fixed integer such that $1 \leqslant n < m$ (respectively, $2 \leqslant n < m$). Then $\tilde{M}$ is a complex space form if and only if for every point $p \in \tilde{M}$ and for every holomorphic 2n-dimensional linear subspace $T$ of $T_p\tilde{M}$ (respectively, for every antiholomorphic n-dimensional linear subspace $T$ of $T_p\tilde{M}$) there exists a 2n-dimensional (respectively, n-dimensional) submanifold $N$ of $\tilde{M}$ which satisfies the classical Codazzi equation and such that $p \in N$ and $T_pN = T$.*

The following definition is essentially due to B. Y. Chen and K. Ogiue in 1974: a Kaehlerian manifold $\tilde{M}$ satisfies the *axiom of coholomorphic $(2k+l)$-spheres* if for each point $p \in \tilde{M}$ and for every $(2k+l)$-dimensional CR-section $T$ of $T_p\tilde{M}$ there exists a totally umbilical submanifold $N$ of $\tilde{M}$ such that $p \in N$ and $T_pN = T$; and they used it to characterize the locally flat spaces.

THEOREM 6.9 (Chen and Ogiue [18]). *A Kaehlerian manifold of dimension $2m > 4$ is locally flat if and only if it satisfies the axiom of coholomorphic $(2k+l)$-spheres for some integers k and l such that $1 \leqslant k < m$, $1 \leqslant l < m$ $(2k+l < 2m)$.*

In this respect we recall that there do not exist totally umbilical proper CR-submanifolds in any positively or negatively curved Kaehler space.

As stated above, S. Tachibana and S. Kashiwada proved that every geodesic hypersphere with normal section $\xi$ in a complex space form is $J\xi$-quasiumbilical. In 1980, B. Y. Chen and one of the authors studied the following corresponding axiom: a Kaehlerian manifold $\tilde{M}$ satisfies the *axiom of $J\xi$-quasiumbilical hypersurfaces* if and only if for each point $p$ of $\tilde{M}$ and for every hyperplane $T$ in $T_p\tilde{M}$ with hyperplane normal $\xi$ there exists a $J\xi$-quasiumbilical hypersurface through $p$ with $T$ as a tangent space at $p$, and together with the result of S. Tachibana and S. Kashiwada obtained the following

THEOREM 6.10 (Chen and Verstraelen [22]). *A Kaehlerian manifold of real dimension $> 4$ satisfies the axiom of $J\xi$-quasiumbilical hypersurfaces if and only if it is a complex space form.*

We mention that this theorem generalizes earlier results in this direction from Tashiro and Tachibana [73] and Vanhecke and Willmore [74]. In 1981 we obtained the following particular case of the previous theorem:

THEOREM 6.11 (Van Lindt and Verstraelen [77]). *A Kaehlerian manifold $\tilde{M}$ of real dimension $> 4$ is locally flat if and only if for any point $p$ of $\tilde{M}$ and for any hyperplane $T$ of $T_p\tilde{M}$ with hyperplane normal $\xi$ there exists a $J\xi$-hypercylinder $N$ in $\tilde{M}$ such that $p \in N$ and $T_pN = T$.*

A similar result also holds for these hypercylinders for which the distinguished tangent section together with $\xi$ spans a totally real plane [75].

Finally, we refer to [75] for two partial complex analogues of the characterizations of conformally flat spaces given in Theorems 5.8 and 5.10. In fact, using totally real subspace $T$ in the axioms being considered there, one may conclude that the Kaehlerian manifolds which satisfy these axioms must be Bochner flat.

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*Reçu par la Rédaction le 23. 9. 1983*