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SERIES OF ITERATES

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0. Introduction. The subject-matter of this paper pertains to the classical theory of infinite series established by A. L. Cauchy in the 19-th century and further developed by other mathematicians. Also, the technique which is used here is not a product of the modern mathematics and, as a matter of fact, was available even before the principles of the theory of infinite series were established. If some ideas here appear to be new, it can only be surprising.

In the first two sections, series of iterates of a positive function are discussed. Series of iterates constitute a very comprehensive class which contains: geometric series, Dirichlet series, Bertrand series and more general series which appear in the Cauchy integral test. An Ermakoff type test is also proposed for series of iterates.

An equivalent of Cauchy's integral test is discussed in Sections 3 and 4. An application to power series is given there. Also a remark on the well-known Cauchy-Hadamard theorem is made which includes some new cases of convergent as well as divergent series. For these cases the classical test is inconclusive. A fixed point theorem which generalizes the contraction principle for complete metric spaces is presented in Section 5. It is based upon the equivalent of Cauchy's integral test.

1. Series of iterates of a positive function.

1.1. Let be given a real-valued function Q with the property

(a) $0 < Q(s) < s$ for $0 < s \leqslant s_1$.

Then the series

$$S = \sum_{n=1}^{\infty} s_n, \quad \text{where } s_{n+1} = Q(s_n),$$

is called a *series of iterates* (*generated by* Q).

A class of convergent series of iterates is defined in [1] (see also [2]) and it is shown there that the series S is convergent and

$$\sum_{i=n}^{\infty} s_i \leqslant \int_0^{s_n} g(s)$$

if Q also satisfies in addition to (a), the following conditions:

(b) The function $g(s) = s/(s-Q(s))$ is non-increasing.

(c) The inequality

$$\int_0^{s_1} g(s)\,ds < \infty$$

holds true.

If conditions (a) and (b) are satisfied, then condition (c) is also necessary for the convergence of the series S provided that the following condition is satisfied:

(d) There exist positive numbers a and d such that

$$[Q(u)-Q(v)]/(u-v) \geqslant d \quad \text{for all } 0 < v < u < a.$$

In this case the existence of the integrals $\int_a^{s_1} g(s)\,ds$ is obviously required for $0 < a < s_1$.

Condition (d) can be replaced by the following one:

(d′) The function $Q(s)$ is differentiable in an open interval $(0, a)$ and the derivative $Q'(s)$ has a limit as $s \to 0+$.

In fact, since the function $Q(s)/s$ is non-decreasing, by (b) and, consequently, by (a), it has a positive limit not greater than 1. Hence, by de L'Hospital's rule, the derivative $Q'(s)$ has a positive limit as $s \to 0+$. This implies that condition (d) is satisfied.

1.2. *Let $y = f(s)$ be a non-decreasing function satisfying the condition $0 < f(s) < 1$ for $0 < s \leqslant s_1$. Put $Q(s) = s(1-f(s))$ for $0 < s \leqslant s_1$ and $s_{n+1} = Q(s_n)$, $n = 1, 2, \ldots$ If*

$$\int_0^{s_1} [f(s)]^{-1} ds < \infty,$$

then

$$\sum_{n=1}^{\infty} s_n < \infty \quad \textit{and} \quad \sum_{i=n}^{\infty} s_i \leqslant \int_0^{s_n} [f(s)]^{-1} ds.$$

In fact, it is easily seen that conditions (a), (b), and (c) are satisfied with $g(s) = [f(s)]^{-1}$.

As a particular case, we obtain the following convergent series of iterates by putting $f(s) = s^a/(1+a)$ for $0 < s \leqslant s_1 < 1$ and $0 < a < 1$:

$$s_{n+1} = Q(s_n), \quad \text{where } Q(s) = s\big(1 - s^a/(1+a)\big)$$

(see [2]).

1.3. *Let $y = f(s)$ be a non-decreasing function satisfying the condition $0 < f(s) < 1$ for $0 < s \leqslant s_1$. Assume that the derivative $f'(s)$ exists in some open interval $(0, a)$ and it has a limit as $s \to 0+$. Then the series of iterates $s_{n+1} = Q(s_n)$, where $Q(s) = s\big(1 - f(s)\big)$, is convergent if and only if $\int_0^{s_1} [f(s)]^{-1} ds < \infty$ provided that the integrals $\int_\beta^{s_1} [f(s)]^{-1} ds$ exist for $0 < \beta < s_1$.*

In fact, it is easily seen that conditions (a), (b) and (d′) are satisfied and, therefore, the assertion follows.

1.4. *Suppose that Q is a function satisfying condition* (a). *Suppose, in addition, that Q is differentiable and the derivative $Q'(s)$ has a limit as $s \to 0+$ and satisfies the condition*

(e) $Q'(s) < Q(s)/s$ *for* $0 < s \leqslant s_1$.

Then the series of iterates $s_{n+1} = Q(s_n)$, $n = 1, 2, \ldots$, is convergent if and only if

$$\int_0^{s_1} g(s)\,ds < \infty$$

provided that the integrals $\int_\beta^{s_1} g(s)\,ds$ exist for $0 < \beta < s_1$.

In fact, condition (d′) is satisfied, and so is condition (b), since the derivative of $g(s) = s/\big(s - Q(s)\big)$ exists and is negative, by virtue of (e). It is easy to see that the function

$$Q(s) = s[1 - s^a/(1+a)] \quad \text{for } 0 < s \leqslant s_1 < 1$$

satisfies the above-mentioned hypotheses.

1.5. The Dirichlet series

$$S = \sum_{n=1}^{\infty} 1/n^a, \quad 1 < a,$$

is a series of iterates generated by $Q(s) = (s^{-1/a}+1)^{-a} < s$ (see [2]). The function

$$g(s) = [1 - Q(s)/s]^{-1} = [1 - 1/(1+s^{1/a})]^{-1}$$

is decreasing. The integral in (c) is

$$\int_0^{s_1} g(s)\,ds = \int_0^{s_1} [1 - 1/(1+s^{1/a})]^{-1} ds = a\int_0^{\infty} \{x^{1+a}[1 - (x/(x+1))^a]\}^{-1} dx.$$

A general class of series of iterates can be defined by

$$S = \sum_{n=1}^{\infty} f(n),$$

where f is a positive decreasing continuous function such that $f(x) \to 0$ as $x \to \infty$. In this case we have (see [2])

$$Q(s) = f\big(f^{-1}(s)+1\big), \quad s = f(x), \quad s_n = f(n),$$
$$g(s) = s\big[s - f\big(f^{-1}(s)+1\big)\big]^{-1}.$$

If the function with values $f(x)\,[f(x)-f(x+1)]^{-1}$ is not decreasing, then g is not increasing and condition (b) holds true. If, in addition, f is differentiable, then condition (c) can be written in the form

$$\int_0^{s_1} g(s)\,ds = \int_0^{s_1} \big[s - f\big(f^{-1}(s)+1\big)\big]^{-1} ds$$
$$= \int_1^{\infty} f(x)\,|f'(x)|\,[f(x)-f(x+1)]^{-1} dx < \infty.$$

Condition (d) for Q can be written in the form

$$[f(x+1)-f(y+1)]/[f(x)-f(y)] \geqslant d > 0$$

for sufficiently large $y > x$. This condition can be replaced by the following one:

$f'(x+1)/f'(x)$ has a limit as $x \to \infty$, provided that the function f is differentiable and $f'(x) \neq 0$.

The Bertrand series (see [4])

$$\sum_n [n l_1 n l_2 n \ldots l_{r-1} n \cdot (l_r n)^p]^{-1},$$

where

$$l_0 x = x, \; l_1 x = \log x, \ldots, l_r x = \log l_{r-1} x,$$

provide an important class of convergent series of iterates if $p > 1$. This is a subclass of the series of the form $\sum_n f(n)$ mentioned above.

1.6. *Let Q be a function which satisfies condition* (a) *and put*

$$Q_1(x) = Q(x), \quad Q_2(x) = Q\big(Q(x)\big), \ldots,$$
$$Q_{k+1}x = Q\big(Q_k(x)\big) \quad \textit{for } k = 1, 2, \ldots,$$

and let $\{b_k\}$ be an increasing sequence of positive integers such that $b_{k+1} - b_k \leqslant M(b_k - b_{k-1})$. Then the series

$$\sum_{n=1}^{\infty} s_n, \quad \textit{where } s_{n+1} = Q(s_n),$$

and the series

$$\sum_{k=1}^{\infty}(b_{k+1}-b_k)Q_{b_k}(s_1)$$

are either both convergent or both divergent.

This fact results immediately from Cauchy's condensation test (see [3]), since, by definition, $s_n = Q_n(s_1)$.

2. An Ermakoff type test for series of iterates.

2.1. The argument used by Ermakoff (see [3], p. 296) can be applied to series of iterates in order to obtain a similar test. For this purpose we investigate the convergence of the improper (in general) integral in (c). Let Q be continuous and let (a) be satisfied. Then we obtain

$$\int_0^1 g(s)\,ds = \int_0^{\infty} e^{-x}g(e^{-x})\,ds = \int_0^{\infty} f(x)\,dx = I,$$

$$\text{where } f(x) = e^{-x}g(e^{-x}),\ s = e^{-x}.$$

Put $E(x) = e^x f(e^x)/f(x)$. It follows from the argument of Ermakoff (see [3], p. 297) that the integral I is convergent if there exists a positive number $\theta < 1$ such that $E(x) \leqslant \theta$ for all sufficiently large x, and I is divergent if $E(x) \geqslant 1$ for all sufficiently large x. In terms of g and $s = e^{-x}$ we obtain

$$E(x) = e^{-1/s}g(e^{-1/s})/s^2 g(s) = \mathscr{E}(s)$$

or

$$\mathscr{E}(s) = e^{-2/s}\big(s - Q(s)\big)/s^3\big(e^{-1/s} - Q(e^{-1/s})\big).$$

Thus, the integral I in (c) is convergent if $\mathscr{E}(s) \leqslant \theta < 1$ for all sufficiently small positive s, and I is divergent if $\mathscr{E}(s) \geqslant 1$ for all sufficiently small positive s.

Let us observe that if

$$g(e^{-1/s})/g(s) \quad \text{or} \quad \big(s - Q(s)\big)/\big(e^{-1/s} - Q(e^{-1/s})\big)$$

is bounded, then $\mathscr{E}(s) \leqslant \theta < 1$ for all sufficiently small positive s, since

$$e^{-1/s}/s^2 \to 0 \quad \text{and} \quad e^{-2/s}/s^3 \to 0 \quad \text{as } s \to 0+.$$

2.2. The following Ermakoff type test for series of iterates can now be easily obtained:

Let Q be a continuous function satisfying conditions (a), (b), *and* (d) *or* (d′). *Then the series of iterates generated by Q is convergent if there exists a positive number $\theta < 1$ such that $\mathscr{E}(s) \leqslant \theta$ for all sufficiently small positive s, and this series is divergent if $\mathscr{E}(s) \geqslant 1$ for all sufficiently small positive s.*

The proof follows immediately from the argument of 1.1. This test can also be applied to some special cases considered above.

Remark. The function e^x in Ermakoff's test can of course be replaced by other functions (see [3], p. 298). If $\varphi(x)$ is any monotone increasing positive function, everywhere differentiable, for which $\varphi(x) > x$ always, then we can replace

$$E(x) = e^x f(e^x)/f(x)$$

by

$$E(x) = \varphi'(x) f\big(\varphi(x)\big)/f(x).$$

Let $\varphi(x_0) = 1$ and put $s = 1/\varphi(x)$. Then we obtain

$$\int_0^1 g(s)\,ds = \int_{x_0}^{\infty} \varphi'(x)\,[\varphi(x)]^{-2} g\big(1/\varphi(x)\big)\,dx.$$

Hence, $E(x)$ for the integrand function is

$$E(x) = \frac{\varphi'\big(\varphi(x)\big)\big[\varphi\big(\varphi(x)\big)\big]^{-2} g\Big(1/\varphi\big(\varphi(x)\big)\Big)}{[\varphi(x)]^2 g\big(1/\varphi(x)\big)}.$$

In terms of s we obtain

$$\mathscr{E}(s) = \frac{\varphi'(1/s)\, g\big(1/\varphi(1/s)\big)}{s^2[\varphi(1/s)]^2 g(s)}.$$

Applications of series of iterates are given in [1] and [2].

2.3. *Consider the power series*

$$\sum_{n=0}^{\infty} s_n x^n, \quad \textit{where } s_{n+1} = Q(s_n) \textit{ for } n = 0, 1, \ldots$$

Suppose that the function Q satisfies conditions (a) *and* (b) *of Section* 1.1. *Then the radius of convergence of the power series is* 1 *or the series* $\sum_{n=0}^{\infty} s_n$ *can be majorized by a geometric series.*

Proof. It follows from (a) and (b) that the sequence of $s_{n+1}/s_n = Q(s_n)/s_n < 1$ is non-decreasing and, therefore, it has a limit $\mu \leqslant 1$. If $\mu < 1$, then $s_n \leqslant s_0 \mu^n$ for all n. If $\mu = 1$, then, by virtue of a theorem of Cauchy, $s_n^{1/n} \to \mu = 1$ as $n \to \infty$. Hence, the radius of convergence equals $1/\mu = 1$.

2.4. Abel's limit theorem (see [3]) can be applied in the following way.

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} s_n x^n,$$

where $s_{n+1} = Q(s_n)$ *with* Q *satisfying conditions* (a) *and* (b). *Suppose that*

(i) $\lim_{s\to 0+} Q'(s) = 1.$

Then $\lim_{x\to 1-0} f(x)$ *exists and is equal to* $\sum_{n=0}^{\infty} s_n < \infty$ *if and only if*

$$I = \int_0^{s_0} g(s)\,ds < \infty;$$

and

$$\lim_{x\to 1-0} f(x) = \infty$$

if and only if $I = \infty$, *where* $g(s) = s/(s-Q(s))$. *Condition* (i) *can be replaced by condition* (d) *and* $\sup[Q(s)/s] = 1$.

Proof. Since 1 is the radius of convergence for the power series $f(x)$, the proof follows from Abel's limit theorem and from the fact that, in this case, condition (c) is necessary and sufficient for the convergence of I.

Example. $Q(s) = s(1-s^a/(1+a))$ for $0 < a < 1$ and $0 < s \leqslant s_0 < 1$ satisfies all the hypotheses.

3. Cauchy's integral test.

3.1. *Let* f *be a continuous non-increasing function which is positive for* $x > 0$. *Then, by Cauchy's integral test, the series* $\sum_{n=0}^{\infty} f(n)$ *is convergent if and only if*

$$\int_0^{\infty} f(x)\,dx < \infty.$$

Now, let $B(s)$ *be a non-decreasing continuous positive function defined on the interval* $(0, 1]$ *and let* $q < 1$ *be an arbitrary fixed positive number.*

Then the series $\sum_{n=1}^{\infty} B(q^n)$ *is convergent if and only if*

$$\int_0^1 s^{-1}B(s)\,ds < \infty. \tag{3.1}$$

This fact follows immediately from Cauchy's integral test by putting $f(x) = B(q^x)$. Thus, putting $s = q^x$ we obtain

$$\int_0^{\infty} f(x)\,dx = |\log q|^{-1}\int_0^1 s^{-1}B(s)\,ds.$$

For any function $f(x)$ which satisfies the assumptions made above for Cauchy's integral test we can always define a function $B(s)$ by the formula $f(x) = B(q^x)$ for $s = q^x$, where q is arbitrary with $0 < q < 1$. The function B has, of course, all the properties required above. In this simple way we obtain an equivalent form of Cauchy's integral test.

3.2. *Consider the power series* $\sum_{n=0}^{\infty} a_n x^n$. *Let* $B(s)$ *be a positive non-decreasing continuous function defined on* $[0, 1]$ *and such that*

$$\int_0^1 s^{-1} B(s)\, ds < \infty.$$

Suppose that ϱ *and* $q < 1$ *are positive numbers such that*

$$|a_n| \leqslant B(q^n)/\varrho^n$$

for almost all positive integers n.

Then the power series $\sum_{n=0}^{\infty} a_n x^n$ *is uniformly convergent for* $|x| \leqslant \varrho$.

This fact follows immediately from the equivalent Cauchy integral test.

It turns out that, by using the well-known Cauchy-Hadamard theorem in order to find the radius of convergence for the power series considered in this case, one can only prove the uniform convergence for $|x| \leqslant \varrho$, where $0 < \bar{\varrho} < \varrho$.

3.3. An Ermakoff type test in terms of the function $B(s)$ can easily be established. In other words, we have to find a convergence test for the integral

$$I = \int_0^1 s^{-1} B(s)\, ds.$$

Replacing $g(s)$ in 2.1 by the function $s^{-1}B(s)$ we obtain the required expression for $\mathscr{E}(s)$:

$$\mathscr{E}(s) = e^{-1/s} B(e^{-1/s})/sB(s), \qquad 0 < s \leqslant 1.$$

Hence, the integral I is convergent if there exists a positive number $\theta < 1$ such that

$$\mathscr{E}(s) \leqslant \theta < 1 \tag{3.2}$$

for all sufficiently small positive s, and the integral I is divergent if

$$\mathscr{E}(s) \geqslant 1 \tag{3.3}$$

for all sufficiently small $s > 0$.

Notice that if $B(e^{-1/s})/B(s)$ is bounded for $0 < s \leqslant 1$, then $\mathscr{E}(s) \leqslant \theta < 1$ for sufficiently small $s > 0$, since $e^{-1/s}/s \to 0$ as $s \to 0+$.

There is a more direct approach by investigating the convergence of the integral

$$I_1 = \int_0^\infty B(q^x)\, dx,$$

where $q < 1$ is positive. Then, by Ermakoff's test, we infer that if there exists a positive $\theta < 1$ such that

$$E_q(x) = e^x B(q^{e^x})/B(q^x) \leqslant \theta < 1 \tag{3.4}$$

for all sufficiently large x, then the integral I_1 is convergent, and if

$$E_q(x) \geqslant 1 \tag{3.5}$$

for all sufficiently large x, then the integral I_1 is divergent.

Since the convergence of the integral I_1 for a particular positive $q < 1$ implies the convergence of the integral I, and the convergence of I yields the convergence of I_1 for arbitrary positive $q < 1$, it follows that the convergence (divergence) of I_1 for a particular positive $q < 1$ implies the convergence (divergence) of I_1 for arbitrary positive $q < 1$.

Suppose now that there exist positive numbers $q < 1$ and $\theta < 1$ such that one of the conditions (3.1), (3.2), and (3.4) is satisfied, and assume that $|a_n| \leqslant B(q^n)/\varrho^n$ for almost all positive integers n. Then the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for $|x| \leqslant \varrho$. On the other hand, if there exist positive numbers ϱ and $q < 1$ such that one of the conditions (3.3), (3.5), and $I = \infty$ is satisfied, and $|a_n| \geqslant B(q^n)/\varrho^n$ for almost all positive integers n, then the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely divergent for $|x| \geqslant \varrho$.

Notice that the latter case does not provide information needed to compute the radius of convergence on the basis of the Cauchy-Hadamard theorem.

3.4. *Consider now the Taylor series*

$$\sum_{n=0}^{\infty} (1/n!) f^{(n)}(a)(x-a)^n$$

for the function f. Let $B(s)$ be a continuous non-decreasing function which is positive for $0 < s \leqslant 1$ and satisfies condition (3.1). *Suppose that there exist positive numbers ϱ and $q < 1$ such that*

$$(1/n!)\,|f^{(n)}(a)| \leqslant B(q^n)/\varrho^n$$

for almost all positive integers n. Then the Taylor series for f is absolutely uniformly convergent if $|x-a| \leqslant \varrho$.

Of course, condition (3.1) can be replaced by condition (3.2) or (3.4).

Suppose now that the integral

$$I = \int_0^1 s^{-1} B(s)\,ds = \infty$$

and $\varrho > 0$, $0 < q < 1$ *are such that*

$$(1/n!)\,|f^{(n)}(a)| \geqslant B(q^n)/\varrho^n$$

for almost all positive integers n. *Then the Taylor series for* f *is absolutely divergent if* $|x-a| \geqslant \varrho$.

Of course, the assumption $I = \infty$ can be replaced by condition (3.3) or (3.5).

4. A generalization of the Cauchy-Hadamard theorem.

4.1. It is shown in Section 3.3 that in some cases of convergent as well as of divergent power series the test based on the Cauchy-Hadamard theorem is inconclusive. By using the result of Section 3.3, the Cauchy-Hadamard theorem can be generalized so as to include the cases just mentioned.

Let $\sum_{n=0}^{\infty} a_n x^n$ be a given power series. Denote by B_c the class of all continuous non-decreasing functions which are positive for $0 < s \leqslant 1$ and such that

$$\int_0^1 s^{-1} B(s)\,ds < \infty.$$

Let us denote by B_d the same class of functions for which

$$\int_0^1 s^{-1} B(s) = \infty.$$

Thus, if $B \in B_c$, then the series $\sum_{n=0}^{\infty} B(q^n)$ is convergent for arbitrary positive $q < 1$, and if $B \in B_d$, then the series $\sum_{n=0}^{\infty} B(q^n)$ is divergent for arbitrary positive $q < 1$. Put

$$\mu = \limsup_n |a_n|^{1/n}.$$

It is easy to see that if $B \in B_c \cup B_d$, then

$$\liminf_n \{[B(q^n)]^{1/n}/|a_n|^{1/n}\} \leqslant 1/\mu. \tag{4.1}$$

In fact, the sequence $\{B(q^n)\}$ is non-increasing and positive. Hence, it has a limit $\mathscr{C}$.

If $\mathscr{C} > 0$, then, by a theorem of Cauchy (see [3]),

$$[B(q^n)]^{1/n} \to 1 \quad \text{as } n \to \infty.$$

In this case, inequality (4.1) becomes an equality.

If $\mathscr{C} = 0$, then for almost all positive integers n we have

$$[B(q^n)]^{1/n} \leqslant 1$$

and, therefore, inequality (4.1) holds true.

4.2. (i) *If there exist a function $B \in B_c$ and positive numbers ϱ and $q < 1$ such that*

$$(4.2) \qquad |a_n| \leqslant B(q^n)/\varrho^n$$

or almost all positive integers n, then the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is absolutely convergent if $|x| \leqslant \varrho$.

(ii) *If $\varrho > 1/\mu$ or if there exist a function $B \in B_d$ and positive numbers ϱ and $q < 1$ such that*

$$(4.3) \qquad |a_n| \geqslant B(q^n)/\varrho^n$$

for almost all positive integers n, then the power series $f(x)$ is absolutely divergent if $|x| \geqslant \varrho$.

(iii) *The following equalities hold:*

$$\alpha = \sup_{B \in B_c} \sup_{0<q<1} \liminf_n \{[B(q^n)]^{1/n}/|a_n|^{1/n}\} = 1/\mu,$$

$$\beta = \sup_{B \in B_d} \sup_{0<q<1} \liminf_n \{[B(q^n)]^{1/n}/|a_n|^{1/n}\} = 1/\mu.$$

Proof. Statements (i) and (ii) are discussed in Section 3.3. In order to prove (iii) let us observe that the function $\bar{B}(s) = s$ belongs to B_c and we have

$$\bar{B}(q^n) = q^n \quad \text{and} \quad \alpha_{\bar{B}} = \sup_{0<q<1} \liminf_n \{q/|a_n|^{1/n}\} = 1/\mu.$$

But it follows from (4.1) that $\alpha \leqslant 1/\mu$.

Since $\beta \leqslant 1/\mu$, by virtue of (4.1), the second equality follows from the fact that the function $\bar{B}(s) \equiv 1$ for $0 < s \leqslant 1$ belongs to B_d. Therefore, we have $\bar{B}(q^n) = 1$ for all n and $\beta = 1/\mu$.

Notice that if $\varrho < 1/\mu$, then there exist a function $B \in B_c$ and a positive number $q < 1$ which satisfy (4.2). In fact, we can choose a positive q such that $\mu\varrho < q < 1$ or $\mu < q/\varrho$. Then, by the definition of the upper limit, we have

$$|a_n|^{1/n} \leqslant q/\varrho \quad \text{or} \quad |a_n| \leqslant q^n/\varrho^n$$

for almost all positive integers n. Thus, condition (4.2) is satisfied for the function $B(s) = s$, $0 < s \leqslant 1$, which, of course, belongs to B_c.

Example. The Dirichlet series provides a classical example of a power series for which the Cauchy-Hadamard theorem is inconclusive.

For consider the power series

$$S = \sum_{n=0}^{\infty} x^n/(n+1)^a, \quad \text{where } a > 1.$$

Since the sequence $\{n^{1/n}\}$ has 1 as its limit, the radius of convergence for S is $1/\mu = 1$. Thus, if $|x| < 1$, then the series S is absolutely convergent, and the Cauchy-Hadamard theorem is inconclusive if $|x| = 1$. But if $|x| = 1$, then the convergence of S follows from the Cauchy integral test.

Using this example it is possible to show that there exists a power series which satisfies the hypotheses of the generalized theorem while the classical Cauchy-Hadamard theorem is inconclusive. In other words, for each positive $q < 1$ there exists a function $B \in B_c$ which satisfies condition (4.2).

Let

$$S = \sum_{n=0}^{\infty} a_n x^n, \qquad \text{where } a_n = 1/(n+1)^a \text{ with } a > 1.$$

We define the function B by the formula $B(s) = f(x)$, where $s = q^x$ and $f(x) = 1/(x+1)^a$ for $0 \leqslant x$. The function B is obviously increasing, continuous, and it is easily seen that

$$\int_0^1 s^{-1} B(s)\, ds = |\log q| \int_0^{\infty} f(x)\, dx < \infty.$$

Thus, $B \in B_c$ and also condition (4.2) is satisfied with $\varrho = 1$, since

$$a_n = 1/(n+1)^a = f(n) = B(q^n) = B(q^n)/1^n \quad \text{for } n = 0, 1, \ldots$$

Hence, the power series S is absolutely convergent if $|x| \leqslant 1$. As we mentioned above, the classical Cauchy-Hadamard theorem is inconclusive if $|x| = 1$ which is precisely the radius of convergence for the power series S.

Finally, let us observe that by introducing an equivalent of the Cauchy integral test we have established an important link between the Cauchy integral test and the radius of convergence for power series which is defined by the Cauchy-Hadamard theorem. Moreover, in some inconclusive cases the Cauchy equivalent test may be applicable. However, if instead of B a function f is given for which Cauchy's integral test is applicable, then $B(q^n)$ in (4.2) and (4.3) can be replaced by $f(n)$, $n = 1, 2, \ldots$

5. A fixed point theorem.

5.1. As an application of the equivalent of Cauchy's integral test we obtain the following fixed point theorem:

Let $F: X \to X$ be a continuous mapping of the complete metric space X into itself. Let $B(s) > 0$ for $0 < s \leqslant a$ be a continuous non-decreasing function such that

$$\int_0^a s^{-1} B(s)\, ds < \infty.$$

Suppose that there exist an element $x \in X$ and a positive number $q < 1$ such that $d(Fx, x) \leqslant a$ and

$$d(F^{n+1}x, F^n x) \leqslant B(q^n d(Fx, x)) \quad \text{for } n = 1, 2, \ldots, \tag{5.1}$$

where $d(\cdot, \cdot)$ is the distance in X. Put $x_{n+1} = Fx_n$ for $n = 1, 2, \ldots$

Then the sequence $\{x_n\}$ converges to a fixed point $x^* = Fx^*$.

Proof. We have, for $m > n$,

$$d(x_{m+1}, x_{n+1}) = d(F^m x, F^n x) \leqslant \sum_{i=n}^{m-1} d(F^{i+1}x, F^i x) \leqslant \sum_{i=n}^{m-1} B(q^i d(Fx, x)).$$

Since the series $\sum_{i=1}^{\infty} B(q^i d(Fx, x))$ is convergent (see Section 3.1), the sequence of elements $x_{n+1} = Fx_n$ is a Cauchy sequence and, therefore, it has a limit x^*. The continuity of F implies that $x^* = Fx^*$.

Remark. Suppose that F is a contraction mapping, that is, there exists a positive constant $q < 1$ such that

$$d(Fx, Fy) \leqslant qd(x, y) \quad \text{for all } x, y \in X.$$

Then, of course, F satisfies condition (5.1) if we put $B(s) = s$ and x is an arbitrary element of X.

Notice that this theorem actually yields an application of the Cauchy integral to the general theory of fixed points. The Cauchy integral test appears in its equivalent form discussed in Section 3.1.

As to the rate of convergence of the sequence of the successive approximations $x_{n+1} = Fx_n$, the following observation is immediate. If F is a contraction mapping, then the convergence is linear, that is, the rate of convergence is the same as that for some geometric series. However, in general, the rate depends essentially on the function B.

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