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LINEARITY IN THE MINKOWSKI SPACE WITH NON-STRICTLY CONVEX SPHERES

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I. A subset L of a metric space $\langle X, \varrho\rangle$ is called *linear* if every subset consisting of three points of L is linear. Linearity is defined by the metric betweenness: a triple $\{p, q, r\}$ is *linear* if one of the points p, q, r lies between the two others, and we say that *q lies between p and r* (writing pqr) provided that

$$\varrho(p, r) = \varrho(p, q)+\varrho(q, r).$$

The metric betweenness in the Euclidean n-space E^n (with the ordinary metric) will be denoted by $E(pqr)$ and in the Minkowski n-space M^n (with the metric defined below) by $M(pqr)$.

The aim of the present paper is to give necessary and sufficient conditions for $M(pqr)$.

II. The metric betweenness has the following properties [1]:

1. symmetry of the outer points (pqr implies rqp);
2. special inner point (if pqr and $p \neq q \neq r$, then neither prq nor qrp);
3. transitivity (pqr and prs are equivalent to pqs and qrs).

Following Blumenthal (op. cit., p. 21) we shall define the n-dimensional Minkowski space M^n by introducing a new metric m into the n-dimensional Euclidean space. The procedure is as follows:

Let Σ be a convex surface of E^n and 0 be a point of $E^n \setminus \Sigma$ such that every ray with initial point 0 intersects Σ in exactly one point and that Σ is symmetric with respect to 0. If x and y are two points of E^n, put $m(x, y) = 0$ if $x = y$, and $m(x, y) = xy/0Pxy$, if $x \neq y$, where Pxy is the point at which the ray with initial point 0 and parallel to the vector $\overrightarrow{xy}$ meets Σ, and xy and $0Pxy$ are the Euclidean distances of the corresponding points.

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(1) L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford 1953, p. 33.

Examining the proof of the triangle inequality for m, given by Blumenthal, one can deduce from it that

a. $M(xyz)$ if and only if $E(PxyPxzPyz)$.

b. Either $Pxy = Pxz = Pyz$ or $Pxy \neq Pxz \neq Pyz \neq Pxy$.

c. If $E(xyz)$, then $Pxy = Pxz = Pyz$.

d. $E(xyz)$ implies $M(xyz)$.

If the surface Σ is strictly convex, i.e., if relation $E(PQR)$ and $P, Q, R \in \Sigma$ imply either $P = Q$ or $Q = R$, then, by virtue of a, b, c, and d, we have

THEOREM 1. *If Σ is strictly convex, then relations $E(xyz)$ and $M(xyz)$ are equivalent.*

III. Consider now the general case of Σ not necessarily strictly convex. We shall establish some properties of the betweenness in M^n and then, basing upon them, we proceed to the characterization of all linear triples in M^n. Properties and characterization will be expressed in terms of the Euclidean geometry.

III.1. *If $\{x, y, z\}$ can be translated in E^n onto $\{p, q, r\}$, then $M(xyz)$ and $M(pqr)$ are equivalent.*

This follows from the fact that every translation in E^n preserves parallelism.

III.2. *If $\{x, y, z\}$ and $\{x', y, z'\}$ are homothetic in E^n with respect to the point y, then $M(xyz)$ and $M(x'yz')$ are equivalent.*

This follows from the identities $Pxy = Px'y$, $Pxz = Px'z'$, $Pyz = Pyz'$, and from the proportionality $xy : xz : yz = x'y : x'z' : yz'$ of the distances.

Denote by $R(a, b)$, where a and b are two distinct points of E^n, the Euclidean ray with the initial point a passing through the point b.

III.3. *If $M(xyz)$ and $x \neq y \neq z$, then the set $L = R(y, z) \cup R(y, x)$ is linear in M^n.*

Proof. Take three points $p, q, r \in L$. In virtue of II.d, each of the sets $R(y, x)$ and $R(y, z)$ is linear in M^n. Hence, in order to prove that $\{p, q, r\}$ is linear, it remains to consider the case when p and q belong to one ray and r to another. Without loss of generality we may assume that $r \in R(y, z)$, $p, q \in R(y, x)$ and $E(yqp)$. Choose a point $x' \in R(y, x)$ and a point $z' \in R(y, z)$ such that $E(ypx')$, $E(yrz')$ and $yx : yz = yx' : yz'$. In view of III.2 we then have $M(x'yz')$. The betweenness $E(x'xy)$ implies $M(x'py)$, and since $M(x'yz')$ was just proved, we infer by the transitivity that $M(pyz')$. This last relation and $M(yrz')$, which follows from $E(yrz')$, imply $M(pyr)$. Finally, $M(pyr)$ and $M(pqy)$, which follows from $E(pqy)$, yields $M(pqr)$.

We shall yet examine linearity of triples of the form $\{p, 0, q\}$, where $p, q \in \Sigma$. Denote by $\varphi(x)$, $x \in \Sigma$, the point of E^n symmetric to x with respect to the point 0. Since Σ was assumed symmetric with respect to 0, $\varphi(x) \in \Sigma$.

III.4. *If $p, q \in \Sigma$, then $M(p0q)$ holds if and only if the segment $[p, \varphi(q)]$ is contained in Σ.*

Proof. The quadruple $\{p, q, \varphi(p), \varphi(q)\}$ is a parallelogram with the centre 0. Denote by P the middle-point of the segment $[q, \varphi(p)]$. Then the vector $\overrightarrow{OP}$ is parallel to the vector $\overrightarrow{pq}$, and $OP = \frac{1}{2}pq$. If $[p, \varphi(q)] \subset \Sigma$, then $[q, \varphi(p)] \subset \Sigma$ and $P \in \Sigma$, and we have

$$m(p, 0)+m(0, q) = 1+1 = 2 = pq/OP = m(p, q).$$

Conversely, if $M(p0q)$ holds, we have, in virtue of II.a, $E(Pp0PpqP0q)$. Moreover, $Pp0 = \varphi(p)$, $P0q = q$ and $0Ppq = \frac{1}{2}pq$. Hence Ppq is the middle-point of the segment $[q, \varphi(p)]$. If $q = \varphi(p)$, then $Ppq = q$ and so $[p, \varphi(q)]$ is a degenerated segment contained in Σ. And if $q \neq \varphi(p)$, then, according to II.b, $\varphi(p) \neq Ppq \neq q \neq \varphi(p)$, and so the segment $[p, \varphi(q)]$ has three different points in common with the surface Σ.

The following generalization of Theorem 1 follows now immediately from III.1, III.3 and III.4:

THEOREM 2. *If Σ is convex (not necessarily strictly convex), then $M(xyz)$ holds if and only if there exist points $x' \in R(y, x)$ and $z' \in R(y, z)$ such that $[\tau(x'), \varphi\tau(z')] \subset \Sigma$, where τ is the translation which transforms y in 0.*

COROLLARY. *Given three points $a, b, c \in E^n$ (not linear in E^n), where $n \geqslant 2$, one can define in E^n a Minkowski's metric in such a way that $M(abc)$ holds.*

Indeed, it suffices to take Σ to be a convex surface with the centre in the point $0 = b$ and containing the segment $[a, \varphi(c)]$.

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