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KOLMOGOROFF CONSISTENCY THEOREM FOR GLEASON MEASURES

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One of the fundamental theorems of the classical probability theory is the Kolmogoroff theorem on the consistent family of Borel measures in finite Cartesian products $\mathop{\mathsf{X}}_{t\in G} \Omega_t$, $G \subset T$, of metric separable complete spaces Ω_t, $t \in T$. The theorem states that every measure μ_G on $\mathop{\mathsf{X}}_{t\in G} \Omega_t$ of such a family is of the form

$$\mu_G(B) = \mu\big(B \times \mathop{\mathsf{X}}_{t\in T-G} \Omega_t\big),$$

where μ is a uniquely defined measure on the σ-field of cylindric sets in $\mathop{\mathsf{X}}_{t\in T} \Omega_t$, and B stands for a Borel set in $\mathop{\mathsf{X}}_{t\in G} \Omega_t$.

It is known that in the non-commutative probability theory the role of Cartesian products of probability spaces is played by tensor products of Hilbert spaces and the role of probability measures is played by states on the lattice of orthogonal projectors of some von Neumann algebra A. In the case $A = B(H)$, where H is a separable Hilbert space, these states are called *Gleason measures*. In [4] von Neumann introduced two kinds of infinite tensor products of Hilbert spaces: $\bigotimes_{t\in T} H_t$ and $\bigotimes_{t\in T}^{D} H_t$, and called them *complete* and *incomplete*, respectively. Since both of these spaces are topologically complete, his terminology has not generally been accepted. The space $\bigotimes_{t\in T}^{D} H_t$ is usually referred to as a *stabilized tensor product* of H_t and $\bigotimes_{t\in T} H_t$, simply, as a *tensor product*.

Nowak showed (see [3]) that conditions of consistency for a family of Gleason measures are not sufficient for existence of a state μ on $\bigotimes_{t\in T}^{D} H_t$ such that

$$\mu_G(P) = \mu\big(P \otimes \bigotimes_{t\in T-G} I_t\big), \tag{1}$$

where P is a projector in $\bigotimes_{t\in G} H_t$ and I_t is the identity in H_t.

In the present paper we give necessary and sufficient conditions for existence of a measure μ on a stabilized tensor product of a countable number of separable Hilbert spaces and satisfying condition (1). For the sake of clarity of our considerations we recall some basic definitions.

Let Γ_0 be a subset of the Cartesian product $\bigtimes_{i=1}^{\infty} H_i$ which consists of sequences $\{x_i\}$ such that

$$\sum_{i=1}^{\infty} |1-\|x_i\||<\infty.$$

In the set Γ_0 we introduce an equivalence relation $\sim$ putting $\{x_i\}\sim\{y_i\}$ if

$$\sum_{i=1}^{\infty} |1-(x_i, y_i)|<\infty.$$

Let D be some equivalence class with respect to $\sim$. Denote an element $\{x_i\}\in D$ by $\bigotimes_{i=1}^{\infty} x_i$ (or by $x_1\otimes x_2\otimes\ldots$) and construct formal finite linear combinations of such elements

$$\sum_{k=1}^{n} a_k \bigotimes_{i=1}^{\infty} x_i^{(k)},$$

where a_k are complex numbers. We identify an element

$$x_1\otimes\ldots\otimes a x_n\otimes x_{n+1}\otimes\ldots$$

with

$$a\cdot x_1\otimes\ldots\otimes x_n\otimes x_{n+1}\otimes\ldots,$$

and an element

$$x_1\otimes\ldots\otimes(x_n+y_n)\otimes x_{n+1}\otimes\ldots$$

with the sum

$$x_1\otimes\ldots\otimes x_n\otimes x_{n+1}\otimes\ldots+x_1\otimes\ldots\otimes y_n\otimes x_{n+1}\otimes\ldots$$

In the set of such combinations of $\bigotimes_{i=1}^{\infty} x_i$ we introduce an inner product

$$\left(\sum_{k=1}^{n}\bigotimes_{i=1}^{\infty} x_i^{(k)}, \sum_{j=1}^{m}\bigotimes_{i=1}^{\infty} y_i^{(j)}\right)=\sum_{k=1}^{n}\sum_{j=1}^{m}\prod_{i=1}^{\infty}(x_i^{(k)}, y_i^{(j)}) \tag{2}$$

and then complete the obtained linear unitary space in the metric determined by the inner product (2).

Definition 1. The space constructed as above is called a *D-stabilized tensor product of Hilbert spaces H_i* and is denoted by $\bigotimes_{i=1}^{\infty}{}^{D} H_i$.

This definition is formulated in the spirit of the paper [4]. The other known definition of the concept involves the notion of inductive limit of inductive systems (see, for instance, [2]).

In [4] von Neumann showed that $\bigotimes_{i=1}^{\infty}{}^{D} H_i$ is a Hilbert space and that it is separable if and only if all the H_i are separable. Moreover, the class D contains a sequence $\{a_i\}$ such that $\|a_i\| = 1$ for every natural i. The set of elements of the form

$$x_1 \otimes \ldots \otimes x_k \otimes a_{k+1} \otimes a_{k+2} \otimes \ldots$$

is linearly dense in $\bigotimes_{i=1}^{\infty}{}^{D} H_i$, and the elements

$$e_1^{i_1} \otimes \ldots \otimes e_k^{i_k} \otimes a_{k+1} \otimes a_{k+2} \otimes \ldots,$$

where $\{e_i^j\}$ is an orthonormal basis in H_i, $e_i^1 = a_i$, form a standard orthonormal basis in $\bigotimes_{i=1}^{\infty}{}^{D} H_i$. Since both the class D and the space $\bigotimes_{i=1}^{\infty}{}^{D} H_i$ are uniquely determined by a given sequence $(a) = \{a_i\}$ we shall also denote $\bigotimes_{i=1}^{\infty}{}^{D} H_i$ by $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$.

Assume that A_i for $i = 1, 2, \ldots$ is a linear bounded operator in H_i such that for every sequence $\{x_i\} \in D$ the sequence $\{A_i x_i\}$ belongs to D and

$$\sum_{i=1}^{\infty} |1 - \|A_i\|| < \infty.$$

Then there exists a unique linear bounded operator $\bigotimes_{i=1}^{\infty} A_i$ such that

$$\bigotimes_{i=1}^{\infty} A_i \Big(\bigotimes_{i=1}^{\infty} x_i\Big) = \bigotimes_{i=1}^{\infty} A_i x_i.$$

In particular, the operators

$$\bigotimes_{i=1}^{n} A_i \otimes \bigotimes_{i=n+1}^{\infty} I_i \quad \text{and} \quad \bigotimes_{i=1}^{n} A_i \otimes \bigotimes_{i=n+1}^{\infty} P_{a_i},$$

where I_i is the identity in H_i, and P_{a_i} is a projection on a one-dimensional subspace of H_i generated by a vector a_i, are well defined in $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ for any finite sequence $A_1, A_2, \ldots, A_n$. It can easily be verified that a projection P_v on the subspace generated by

$$v = e_1 \otimes \ldots \otimes e_n \otimes \bigotimes_{i=n+1}^{\infty} a_i$$

is identical with an operator

$$\bigotimes_{i=1}^{n} P_{e_i} \otimes \bigotimes_{i=n+1}^{\infty} P_{a_i}.$$

Definition 2. By a *Gleason measure* on a separable Hilbert space H we understand a function μ which maps the lattice of projectors of $B(H)$ into the interval $\langle 0, 1\rangle$ so that

(i) $$\mu(I) = 1, \quad \mu(0) = 0,$$

(ii) $$\mu\Big(\sum_{j=1}^{\infty} P_j\Big) = \sum_{j=1}^{\infty} \mu(P_j) \quad \text{for mutually orthogonal } P_j.$$

In [1] Gleason proved that if $\dim H \geqslant 3$, then for a given measure μ there exists a trace-class self-adjoint non-negative operator S_μ such that $\mathrm{Tr}\, S_\mu = 1$ and

$$\mu(P) = \mathrm{Tr}\, S_\mu P \quad \text{for every projector } P.$$

We shall call S_μ a *density operator* of μ. Every Gleason measure μ satisfies the condition of normality:

$$\mu(\sup_j P_j) = \sup_j \mu(P_j) \quad \text{for any increasing net of projectors } P_j.$$

Assume that for every finite set G of natural numbers the Gleason measure μ_G on $\bigotimes_{i\in G} H_i$ is given.

Definition 3. The family of measures μ_G is called *consistent* if

$$\mu_G(P) = \mu_{G\cup F}(P \otimes \bigotimes_{i\in F} I_i),$$

where G and F are arbitrary disjoint finite sets of natural numbers.

LEMMA 1. *Let μ_G be a consistent family of Gleason measures and let G_n be the set of numbers $1, 2, \ldots, n$. Then the sequence of operators*

$$Q_n = S_{G_n} \otimes \bigotimes_{i=n+1}^{\infty} P_{a_i},$$

where S_{G_n} is a density operator of measure μ_{G_n}, converges weakly to some self-adjoint non-negative bounded operator in $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$.

Proof. Observe that Q_n are density operators. Indeed, it follows from the form of Q_n that they are self-adjoint and non-negative. One can easily calculate that $\mathrm{Tr}\, Q_n = \mathrm{Tr}\, S_{G_n} = 1$.

Denote by q_n a Gleason measure corresponding to Q_n. Let E_{G_n} be an arbitrary projector in $\bigotimes_{i=1}^{n} H_i$. Making use of Gleason's theorem one can verify that

$$q_n(E_{G_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i) = \mu_{G_n}(E_{G_n}), \tag{3}$$

$$q_n(E_{G_n} \otimes P_{a_{n+1}} \otimes P_{a_{n+2}} \otimes \ldots) = q_n(E_{G_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i). \tag{4}$$

Consider an arbitrary finite system $v_1, v_2, \ldots, v_r$ of vectors of a standard basis and a linear combination of these vectors

$$v = \sum_{j=1}^{r} a_j v_j, \quad \text{where } \sum_{j=1}^{r} |a_j|^2 = 1.$$

The system being finite, we can assume that all v_j are of the form

$$v_j = e_1^{i_{1j}} \otimes \ldots \otimes e_k^{i_{kj}} \otimes \bigotimes_{i=k+1}^{\infty} a_i.$$

Thus we have

$$P_v = P_{w_{kr}} \otimes P_{a_{k+1}} \otimes P_{a_{k+2}} \otimes \ldots = E_{G_k} \otimes P_{a_{k+1}} \otimes P_{a_{k+2}} \otimes \ldots,$$

where

$$w_{kr} = a_1 e_1^{i_{11}} \otimes \ldots \otimes e_k^{i_{k1}} + \ldots + a_r e_1^{i_{1r}} \otimes \ldots \otimes e_k^{i_{kr}} \quad \text{and} \quad E_{G_k} = P_{w_{kr}}.$$

For $n > k$ using (3) and (4) we have

$$\begin{aligned} 0 &\leqslant (Q_{n+1}v, v) = q_{n+1}(P_v) = q_{n+1}(E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes P_{a_{n+1}} \otimes \bigotimes_{i=n+2}^{\infty} I_i) \\ &= \mu_{G_{n+1}}(E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes P_{a_{n+1}}) \\ &\leqslant \mu_{G_{n+1}}(E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes I_{n+1}) \\ &= \mu_{G_n}(E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n}) \\ &= q_n(E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i) = q_n(P_v) = (Q_n v, v). \end{aligned}$$

Thus, if v is a linear combination of vectors of the standard basis and $\|v\| = 1$, then the sequence $(Q_n v, v)$ is convergent. It is easy to show that this holds for any v being a linear combination of vectors of the standard basis. Since the norms of operators Q_n are mutually bounded, we have the convergence of $(Q_n v, v)$ for any vector v of $\bigotimes_{i=1}^{\infty (a)} H_i$. Using the polarization formula we obtain the convergence of bilinear forms $(Q_n v, f)$ for any $v, f \in \bigotimes_{i=1}^{\infty (a)} H_i$. Thus the sequence Q_n is weakly convergent to some bounded linear operator S, which is evidently self-adjoint and non-negative.

LEMMA 2. *If for a consistent family of Gleason measures there exists on $\bigotimes_{i=1}^{\infty (a)} H_i$ a Gleason measure μ which satisfies* (1), *then $Q_n \rightharpoonup S$, where S*

is the density operator of the measure μ, and Q_n are the operators defined as above.

Proof. Using (1) and (3) we have

$$q_n(E_{G_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i) = \mu(E_{G_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i).$$

For $n \geqslant k$ we put

$$P_n = E_{G_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i.$$

$\{P_n\}$ is a decreasing sequence of projectors with a limit

$$P = E_{G_k} \otimes \bigotimes_{i=k+1}^{\infty} P_{a_i}.$$

Thus P is the infimum of $\{P_n\}$ and we have

$$\lim_{n\to\infty} \mu(P_n) = \mu(P). \tag{5}$$

Taking v such as in the proof of Lemma 1 we can estimate

$$|(Q_n v, v) - (Sv, v)| = |q_n(P_v) - \mu(P_v)| = |\mu(P_n) - \mu(P)| < \varepsilon \quad \text{for } n > N,$$

which exists by (5). By the same reasoning as in Lemma 1 we obtain $Q_n \rightharpoonup S$.

THEOREM 1. *For a given consistent family of Gleason measures (μ_G) there exists its extension on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ (i.e., a measure μ for which* (1) *holds) if and only if one of the following two equivalent conditions is satisfied:*

$$Q_n \rightharpoonup S, \quad \textit{where } \operatorname{Tr} S = 1, \tag{A}$$

$$\sum_k \sum_{\{i_1,\ldots,i_k\}} \lim_{n\to\infty} (S_{G_n} e_1^{i_1} \otimes \ldots \otimes e_k^{i_k} \otimes a_{k+1} \otimes \ldots \otimes a_n, \\ e_1^{i_1} \otimes \ldots \otimes e_k^{i_k} \otimes a_{k+1} \otimes \ldots \otimes a_n) = 1. \tag{B}$$

Proof. It is easy to see that $\operatorname{Tr} S$ is equal to the sum in (B). Thus (A) $\Rightarrow$ (B). By Lemma 1 we have (B) $\Rightarrow$ (A). The necessity of condition (A) is evident by Lemma 2. Thus we must prove only the sufficiency of (A).

Assume that (A) is satisfied. Then S is a density operator of some measure μ. We show that for every projector E_{G_k} in $\bigotimes_{i=1}^{k} H_i$

$$\mu(E_{G_k} \otimes \bigotimes_{i=k+1}^{\infty} I_i) = \mu_{G_k}(E_{G_k}). \tag{6}$$

We have, for some orthonormal basis $\{v_j\}$,

$$(6')\quad \mu(E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_i)=\mathrm{Tr}(S\cdot E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_i)$$
$$=\sum_j(S\cdot E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_iv_j,v_j)=\sum_j\lim_{n\to\infty}(Q_n\cdot E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_iv_j,v_j)$$
$$\leqslant\lim_{n\to\infty}\mathrm{Tr}(S_{G_n}\cdot E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_i)=\lim_{n\to\infty}\mu_{G_n}(E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_i)=\mu_{G_k}(E_{G_k}).$$

Similarly,

$$(6'')\quad \mu\big((\bigotimes_{i=1}^{k}I_i-E_{G_k})\otimes\bigotimes_{i=k+1}^{\infty}I_i\big)\leqslant\mu_{G_k}(\bigotimes_{i=1}^{k}I_i-E_{G_k}).$$

But

$$\mu(E_{G_k}\otimes\bigotimes_{i=k+1}^{\infty}I_i)+\mu\big((\bigotimes_{i=1}^{k}I_i-E_{G_k})\otimes\bigotimes_{i=k+1}^{\infty}I_i\big)=1$$
$$=\mu_{G_k}(E_{G_k})+\mu_{G_k}(\bigotimes_{i=1}^{k}I_i-E_{G_k}),$$

which proves the equality in (6′) and (6″). Thus condition (6) is proved, and so is Theorem 1.

THEOREM 2. *Let a consistent family of Gleason measures* (μ_G) *have an extension* $\mu^{(a)}$ *on* $\bigotimes_{i=1}^{\infty}{}^{(a)}H_i$. *Then it has an extension* $\mu^{(b)}$ *on* $\bigotimes_{i=1}^{\infty}{}^{(b)}H_i$ *if and only if the sequences* $(a)=\{a_i\}$ *and* $(b)=\{b_i\}$ *satisfy the condition*

$$(*)\qquad \sum_{i=1}^{\infty}\big|1-|(a_i,b_i)|\big|<\infty.$$

Proof. If $\sum_{i=1}^{\infty}|1-(a_i,b_i)|$ is also convergent, then $\bigotimes_{i=1}^{\infty}{}^{(a)}H_i$ and $\bigotimes_{i=1}^{\infty}{}^{(b)}H_i$ are the same spaces. Thus assume that $(*)$ holds, but

$$\sum_{i=1}^{\infty}|1-(a_i,b_i)|=\infty.$$

Then (cf. [4]) there exists a sequence $(z)=\{z_i\}$ of complex numbers such that

$$|z_i|=1,\qquad \sum_{i=1}^{\infty}|1-z_i|=\infty,\qquad\text{and}\qquad \bigotimes_{i=1}^{\infty}{}^{(b)}H_i=\bigotimes_{i=1}^{\infty}{}^{(za)}H_i,$$

where $(za)=\{z_ia_i\}$. Assume that the family (μ_G) has an extension $\mu^{(a)}$ on $\bigotimes_{i=1}^{\infty}{}^{(a)}H_i$. Using Theorem 1 we have

$$Q_n^{(a)}=S_{G_n}\otimes P_{a_{n+1}}\otimes P_{a_{n+2}}\otimes\ldots\rightharpoonup S^{(a)},$$

where $S^{(a)}$ is a density operator of $\mu^{(a)}$. We show now that the sequence

$$Q_n^{(za)} = S_{G_n} \otimes P_{z_{n+1}a_{n+1}} \otimes P_{z_{n+1}a_{n+2}} \otimes \dots$$

converges weakly to $S^{(za)}$, where $S^{(za)}$ is a density operator of some measure $\mu^{(za)}$. Observe that $Q_n^{(za)}$ is of the form

$$Q_n^{(za)} = U^{(z)} S U^{(z)-1},$$

where $U^{(z)}$ is an isometry of $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ onto $\bigotimes_{i=1}^{\infty}{}^{(za)} H_i$ given by the formula

$$U^{(z)} \bigotimes_{i=1}^{\infty} x_i = \bigotimes_{i=1}^{\infty} z_i x_i.$$

(The operators $U^{(z)}$ were defined and studied in [4].) Putting

$$S^{(za)} = U^{(z)} S^{(a)} U^{(z)-1},$$

we have

$$\begin{aligned}|(Q_n^{(za)} v, f) - (S^{(za)} v, f)| &= |(U^{(z)} Q_n^{(a)} U^{(z)-1} v, f) - (U^{(z)} S^{(a)} U^{(z)-1} v, f)| \\ &= |(Q_n^{(a)} U^{(z)-1} v,\ U^{(z)-1} f) - (S U^{(z)-1} v,\ U^{(z)-1} f)| < \varepsilon\end{aligned}$$

for $n > N$, where N is chosen for $U^{(z)-1} v$ and $U^{(z)-1} f$ in view of $Q_n^{(a)} \rightharpoonup S^{(a)}$. Since $U^{(z)}$ is a unitary operator, we have

$$\operatorname{Tr} S^{(za)} = \operatorname{Tr} S^{(a)} = 1.$$

Suppose now that for the sequence $\{b_i\}$ condition (*) is not satisfied and on $\bigotimes_{i=1}^{\infty}{}^{(b)} H_i$ and $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ there exist measures $\mu^{(b)}$ and $\mu^{(a)}$, respectively, for which (1) holds. Thus for some projector P of the form

$$P = P_{e_1} \otimes \dots \otimes P_{e_k} \otimes P_{b_{k+1}} \otimes P_{b_{k+2}} \otimes \dots$$

we have $\mu^{(b)}(P) = \eta > 0$. Then, obviously,

$$\mu^{(b)}(P_{e_1} \otimes \dots \otimes P_{e_k} \otimes P_{b_{k+1}} \otimes \dots \otimes P_{b_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i) \geqslant \eta > 0,$$

and hence

$$\mu_{G_n}(P_{e_1} \otimes \dots \otimes P_{e_k} \otimes P_{b_{k+1}} \otimes \dots \otimes P_{b_n}) \geqslant \eta > 0.$$

On the other hand, consider a sequence of projectors $P_n^{(a)}$ in $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ of the form

$$P_n^{(a)} = P_{c_1} \otimes \dots \otimes P_{c_k} \otimes P_{b_{k+1}} \otimes \dots \otimes P_{b_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i.$$

This sequence is decreasing and its limit is 0, but

$$\mu^{(a)}(P_n^{(a)}) = \mu_{Q_n}(P_{e_1} \otimes \ldots \otimes P_{e_k} \otimes P_{b_{k+1}} \otimes \ldots \otimes P_{b_n}) \geqslant \eta > 0$$

which contradicts the normality of $\mu^{(a)}$.

One of the specially important cases of consistent families of measures are product measures. A measure μ on $H_1 \otimes H_2$ is called a *product of measures* μ_1 and μ_2 if, for any projector P of the form $P = P_1 \otimes P_2$,

$$\mu(P) = \mu_1(P_1) \cdot \mu_2(P_2).$$

It is well known that the density operator S of the measure $\mu_1 \otimes \mu_2$ is equal to $S_1 \otimes S_2$. If for the family of product measures there exists its extension on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$, we shall say that there exists a *product measure* $\bigotimes_{i=1}^{\infty}{}^{(a)} \mu_i$. We have the following

THEOREM 3. *A product measure $\bigotimes_{i=1}^{\infty}{}^{(a)} \mu_i$ on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ exists if and only if one of the following two equivalent conditions is satisfied:*

$$\text{(C)} \qquad \sum_{i=1}^{\infty} |1 - (S_i a_i, a_i)| < \infty,$$

where S_i is a density operator corresponding to the measure μ_i,

$$\text{(D)} \qquad \sum_{i=1}^{\infty} |1 - \max_k \lambda_i^k| < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} |1 - |(a_i, e_i^{k_i})|| < \infty,$$

where $\{\lambda_i^k\}_{k=1}^{\infty}$ is the sequence of eigenvalues of an operator S_i, and $e_i^{k_i}$ the is eigenvector of S_i corresponding to the maximal λ_i^k.

Proof. Assume that a measure

$$\mu = \bigotimes_{i=1}^{\infty}{}^{(a)} \mu_i$$

exists. Then for some projector P of the form

$$P = P_{e_1} \otimes \ldots \otimes P_{e_k} \otimes P_{a_{k+1}} \otimes P_{a_{k+2}} \otimes \ldots$$

we have $\mu(P) > 0$. But

$$\mu(P) = \lim_{n \to \infty} \mu(P_n),$$

where

$$P_n = P_{e_1} \otimes \ldots \otimes P_{e_k} \otimes P_{a_{k+1}} \otimes \ldots \otimes P_{a_n} \otimes \bigotimes_{i=n+1}^{\infty} I_i,$$

$$\mu(P_n) = \prod_{i=1}^{k} (S_i e_i, e_i) \cdot \prod_{i=k+1}^{n} (S_i a_i, a_i),$$

$$\lim_{n\to\infty} \mu(P_n) = \prod_{i=1}^{k} (S_i e_i, e_i) \cdot \prod_{i=k+1}^{\infty} (S_i a_i, a_i).$$

Thus

$$\prod_{i=k+1}^{\infty} (S_i a_i, a_i) > 0$$

which is equivalent to (C). Assume that (C) holds. Then

$$S = \bigotimes_{i=1}^{\infty} S_i$$

is a well-defined self-adjoint non-negative operator in $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ and it is a limit of a sequence of operators

$$Q_n = S_1 \otimes \ldots \otimes S_n \otimes P_{a_{n+1}} \otimes P_{a_{n+2}} \otimes \ldots$$

It is easy to calculate that $\operatorname{Tr} S = 1$. Thus, by Theorem 1, our family of product measures has an extension on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$.

One can verify that (C) $\Rightarrow$ (D).

Assume now that (D) holds. Then

$$\sum_{i=1}^{\infty} |1-(S_i e_i^{k_1}, e_i^{k_i})| = \sum_{i=1}^{\infty} |1-\max_k \lambda_i^k| < \infty$$

and the family $(\bigotimes_{i\in G} \mu_i)$ has an extension on $\bigotimes_{i=1}^{\infty}{}^{\{e_i^{k_i}\}} H_i$. From the second part of condition (D) it follows that the family $(\bigotimes_{i\in G} \mu_i)$ has an extension also on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$ (by Theorem 2).

COROLLARY. *If all μ_i are pure states (i.e., their density operators S_i are of the form $S_i = P_{e_i}$), then the measure $\bigotimes_{i=1}^{\infty}{}^{(a)} \mu_i$ exists if and only if*

$$\sum_{i=1}^{\infty} \big|1-|(e_i, a_i)|\big| < \infty.$$

Remark. The consistent family of Gleason measures described in [3] is a family of product measures for which $(S_i a_i, a_i) = 0$ for $i \geqslant 2$. Obviously, this family does not satisfy condition (C) and, therefore, cannot have an extension on $\bigotimes_{i=1}^{\infty}{}^{(a)} H_i$.

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