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WHITTAKER TRANSFORMS ON REAL-RANK ONE LIE GROUPS

BY

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1. Introduction. Let G be a real-rank one semisimple Lie group. Fix a maximal compact subgroup K, a minimal parabolic subgroup $P = NAM$, and a representative $m^* \in K$ for the action of the Weyl group of (G, A). Let γ be a finite-dimensional irreducible representation of P, and let $(\pi_\gamma, \mathcal{H}_\gamma)$ be the induced principal series representation of G.

Let π be an irreducible unitary representation of MN whose restriction to N is non-trivial. In this paper we study the MN *Whittaker transform* of C^∞ functions $\phi \in \mathcal{H}_\gamma$. This transform is (formally) defined by the operator-valued integral

$$W_\pi(\phi) = \int_N \phi(m^* n)\pi(n)\, dn\,. \tag{1.1}$$

The main problems that we address are the convergence of this integral (in a suitable operator topology), its analytic continuation (as a function of the A parameter of γ), and the functional equation it satisfies (arising from the intertwining operator between π_γ and $\pi_{m^*\gamma}$).

To solve these problems, we consider general operators $\pi(f)$ defined by the sesquilinear form

$$\int_N f(n)\langle \pi(n)\xi, \eta\rangle\, dn\,, \tag{1.2}$$

where $f \in L^1_{\mathrm{loc}}(N)$ but is not necessarily integrable at infinity. By general results of Howe–Moore [HM79], one knows that for some large value of p and a dense set of vectors ξ, η, the matrix entry functions $\langle \pi(n)\xi, \eta\rangle$ are in $L^p(N)$ modulo the projective kernel of π. This suggests that $\pi(f)$ can exist, at least as an unbounded operator, even if f does not decay quite rapidly enough to be integrable at infinity. This is the case for A-homogeneous functions (e.g. the kernels for the Kunze–Stein intertwining operators), which can never be integrable both at 1 and at infinity.

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The following question then arises: If $g \in L^1(N)$ is such that the operators $\pi(f)$ and $\pi(f * g)$ exist in a generalized sense, then does one have

$$\pi(f * g) = \pi(f)\pi(g) \tag{1.3}$$

(here $*$ denotes convolution on N) as operators on some dense domain? The validity of this equation is the key point in obtaining a functional equation for the Whittaker transform.

Our results are closely related to those of Schiffmann [Sch71], who considered only irreducible representations of N and did not explicitly use the decay of the matrix entries of the representations. The present paper owes a major debt to Schiffmann's work, although our basic strategy for defining the "generalized Fourier transform" $\pi(f)$ and proving (1.3) is different.

Our interest in extending Schiffmann's results to allow representations of MN rather than just N comes from the fact that (MN, M) is a Gelfand pair ([Kor82]; cf. [BJR89]). As shown by Faraut [Far82], the spherical Fourier analysis on this pair can be applied to harmonic analysis on the Riemannian symmetric space G/K via the Iwasawa decomposition $G = NAK$. Faraut's approach emphasizes the differential equations satisfied by spherical functions. The representation-theoretic approach of this paper provides an alternative derivation of the functional equation and analytic continuation for Faraut's M-spherical Whittaker vectors.

The paper is organized as follows: In Section 3 we describe models for the infinite-dimensional irreducible representations π of MN and in Section 4 we obtain decay estimates for matrix entry functions when ξ, η are differentiable vectors for π (Theorems 4.1 and 4.6). We use these estimates in Section 5 to prove that integral (1.2) is absolutely convergent when f is locally integrable and satisfies a weak decay condition at infinity (expressed in terms of the homogeneous gauge on N). The operator $\pi(f)$ although unbounded on $\mathcal{H}(\pi)$, is continuous from the space of C^r vectors for π to its dual, for an explicit value of r (cf. Theorems 5.1 and 5.2).

We apply this general result in two situations. First, in Section 6 where f is an A-homogeneous function on N obtained from a Schwartz function on N by Fourier transform along A orbits. Then in Section 7 we prove that integral (1.1) converges absolutely (as a sesquilinear form), for λ in a half-plane which includes the imaginary axis. Following Schiffmann (*loc. cit.*), we obtain the analytic continuation of this integral in the A parameter of γ.

Finally, in Section 8 we prove that equation (1.3) holds for f a homogeneous kernel and g a smooth principal series function.

When $G = \mathrm{SL}(2, \mathbf{R})$, the irreducible representations of MN are one-dimensional and our methods based on decay of matrix entries do not apply. The Whittaker transform in this case is described in [Far82, §I]. In this paper we shall assume that $\dim G/K > 2$. For references to the literature on Whittaker transforms associated with one-dimensional representations of N for general semisimple groups and parabolic subgroups, cf. [Mat88]. We thank Jacek Dziubański for correcting some inaccuracies in a preliminary version of this paper.

Notation. We denote the real numbers, complex numbers, quaternions, and octonions (Cayley numbers) by $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$, respectively. If $\mathbf{F}$ is one of these division algebras, then $\Im z$ denotes the imaginary part of an element $z \in \mathbf{F}$. If $z \in \mathbf{C}$, then $\Re z$ denotes the real part of z. If $x \in \mathbf{R}$, then $\lfloor x \rfloor$ denotes the largest integer $n \leq x$.

2. Differentiable vectors. Our treatment of Whittaker transforms will make essential use of scales of spaces of differentiable vectors for a representation. We recall the following notation and results [Goo69], [Goo70].

Let G be any Lie group with Lie algebra $\mathbf{g}$, and let $U(\mathbf{g})$ be the universal enveloping algebra of the complexification of $\mathbf{g}$. Let π be a strongly continuous representation of G on a Banach space $\mathcal{H}(\pi)$. Let $\mathcal{H}^r(\pi)$ be the space of r times differentiable vectors for π with norm $\|v\|_r$. These spaces are initially defined for positive integers r in terms of strong differentiability of the vector-valued function $g \mapsto \pi(g)v$, and for negative r by duality. These spaces are G-invariant and G acts boundedly, relative to the norm $\|v\|_r$. Let

$$\mathcal{H}^\infty(\pi) = \bigcap_{r>0} \mathcal{H}^r(\pi)$$

be the space of C^∞ vectors for π. There is a canonical representation of $U(\mathbf{g})$ on $\mathcal{H}^\infty(\pi)$ that we also denote by π.

When π is unitary, then the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}(\pi)$ extends to a continuous sesquilinear pairing between $\mathcal{H}^\infty(\pi)$ and

$$\mathcal{H}^{-\infty}(\pi) = \bigcup_{r\in\mathbf{R}} \mathcal{H}^r(\pi).$$

We use this pairing to identify $\mathcal{H}^{-r}(\pi)$ with the dual space of $\mathcal{H}^r(\pi)$ for all $r \in \mathbf{R}$.

For multiples of the regular representation of a compact group there is the following vector-valued version of the Sobolev lemma.

LEMMA 2.1. *Let M be a compact Lie group. Let $\mathcal{E}$ be a Hilbert space (finite or infinite-dimensional), and let π be the left or right regular representation of M on $L^2(K;\mathcal{E})$. Let r, k be integers such that $r > k + \frac{1}{2}\dim M$. Then there is a continuous embedding*

$$\mathcal{H}^r(\pi) \subset C^k(M;\mathcal{E})$$

(where $C^k(M;\mathcal{E})$ is the space of k times strongly differentiable $\mathcal{E}$-valued functions on M).

Proof. It is enough to consider the case where π is the right regular representation, since the map $m \mapsto m^{-1}$ interchanges the left and right actions. Let R be the right regular representation of M on scalar-valued functions (the case $\dim \mathcal{E} = 1$). From [Wal73, Lemma 5.7.5] we know that if $r > \frac{1}{2}\dim M$ then there is a constant $C_r < \infty$ so that

$$|f(m)| \leq C_r \|f\|_r \tag{2.1}$$

for $f \in \mathcal{H}^r(R)$. Furthermore, f is continuous on M. To extend this result to the vector-valued case, we may assume that $\mathcal{E} = l^2$. Then π is a multiple of $\mathbf{R}$, and $f \in \mathcal{H}^r(\pi)$ corresponds to a sequence $\{f_n\} \subset \mathcal{H}^r(R)$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_r^2 = \|f\|_r^2 < \infty .$$

Suppose $r > \frac{1}{2}\dim M$. Then each f_n is continuous and from (2.1) we have

$$\sum_{n=1}^{\infty} |f_n(m)|^2 \leq C_r \|f\|_r^2$$

for $m \in M$. It follows that the function $m \mapsto \{f_n(m)\}$ is continuous from M to l^2. Iterating this argument for the derivatives of f yields the lemma. ∎

3. Representations of MN. We recall from [KLW77] the construction of explicit models of the irreducible unitary representations of MN. These models furnish oscillatory integral formulas for the matrix entries, to which we will apply stationary-phase estimates.

We begin by recalling the fine structure of the group N (cf. [Hel70], [Sch71], [Cow82], [CK83]). Let B be the Cartan–Killing form on G, let β be the indivisible positive root of $\mathbf{a}$ on $\mathbf{n}$, and let $H \in \mathbf{a}$ be the coroot to β ($\beta(H) = 2$). Define an inner product on $\mathbf{n}$ by

$$\langle X, Y \rangle = -8B(X, \theta Y)/B(H, H), \tag{3.1}$$

where θ is the Cartan involution on G. Denote the norm on $\mathbf{n}$ associated with this inner product by $|\cdot|$. There is an orthogonal splitting

$$\mathbf{n} = \mathbf{v} \oplus \mathbf{z} \tag{3.2}$$

where the adjoint A action is multiplication by a^{β} on $\mathbf{v}$ and multiplication by $a^{2\beta}$ on $\mathbf{z}$. Set $p = \dim \mathbf{v}$ and $q = \dim \mathbf{z}$. By classification of rank-one symmetric spaces, $\mathbf{z} = \mathfrak{I}(\mathsf{F})$, where $\mathsf{F} = \mathsf{R}, \mathsf{C}, \mathsf{H}$, or O. Since $\operatorname{ad} H$ acts by 2 on $\mathbf{v}$ and by 4 on $\mathbf{z}$ the normalizing constant $8/B(H, H)$ in (3.1) is $(p + 4q)^{-1}$.

For $Y \in \mathbf{v}$, $Z \in \mathbf{z}$, define $n(Y, Z) = \exp(2Y + Z)$. For $t > 0$ define

$a_t = \exp(\frac{1}{2}\log(t)H)$ and set

$$\delta_t = \mathrm{Ad}(a_t)|_N .$$

Then δ_t is a group of *dilations* on N, scaling by t on $\mathbf{v}$ and by t^2 on $\mathbf{z}$. Define a *gauge function* $\mathcal{N}$ on N by

$$\mathcal{N}(n(Y,Z)) = (|Y|^4 + |Z|^2)^{1/4} . \tag{3.3}$$

Then $\mathcal{N}(\delta_t n) = t\mathcal{N}(n)$ and

$$\mathcal{N}(nn') \le \mathcal{N}(n) + \mathcal{N}(n') \tag{3.4}$$

for $n, n' \in N$. Set $Q = p + 2q$, the *homogeneous dimension* of N relative to the group of dilations $\{\delta_t \mid t > 0\}$.

Fix a representative m^* for the non-trivial element of the Weyl group $W(\mathbf{a},\mathbf{g})$. The A components in the Iwasawa and Bruhat factorizations of m^*n can be expressed in terms of the norm and gauge functions as follows. Let $Y \in \mathbf{v}$, $Z \in \mathbf{z}$, and $1 \ne n \in N$. Then

$$m^*n(Y,Z) \in N a_t K , \quad t^{-2} = (1 + |Y|^2)^2 + |Z|^2 , \tag{3.5}$$

$$m^*n \in N M a_t \theta(N) , \quad t = \mathcal{N}(n)^{-2} \tag{3.6}$$

(cf. [Sch71, Prop. 2.1]; note that $|X|^2 = 2Q(X)$ in Schiffmann's notation and we are using N in place of $\theta(N)$).

We now describe the holomorphically-induced model for the irreducible representations of N when $\mathbf{z} \ne 0$. To handle the three cases $\mathsf{F} = \mathbf{C}, \mathbf{H}, \mathbf{O}$ in a uniform way, it is convenient to view N as a nilpotent group of *type H* (cf. [KR83]). For $h \in \mathbf{z}$ define a skew-symmetric operator J_h on $\mathbf{v}$ by

$$\langle J_h X, Y\rangle = \langle h, [X,Y]\rangle , \quad X, Y \in \mathbf{v} . \tag{3.7}$$

From the invariance of the Killing form we have $J_h X = [h, \theta X]$. Thus $J_h^2 X = [h, \theta[h,\theta X]] = [[h,\theta h], X] = \beta([h,\theta h])X$, since $[h,X] = 0$ and $[h,\theta h] \in \mathbf{a}$. But

$$\beta([h,\theta h]) = 2B(H,[h,\theta h])/B(H,H) = 2B([H,h],\theta h)/B(H,H) .$$

Since $[H,h] = 2\beta(H)h = 4h$, we see that

$$J_h^2 X = -|h|^2 X .$$

(The normalizing factor in (3.1) is determined by this relation.) Thus every non-zero vector $h \in \mathbf{z}$ defines a *complex structure* $\widetilde{J}_h = |h|^{-1} J_h$ on $\mathbf{v}$ and a Hermitian inner product

$$\langle X, Y\rangle_h = |h|\langle X, Y\rangle - i\langle J_h X, Y\rangle . \tag{3.8}$$

Let $|X|_h = |h|^{1/2}|X|$ be the norm associated with this inner product.

Let $h \in \mathbf{z} \setminus \{0\}$. A smooth function f on $\mathbf{v}$ is *h-holomorphic* if $df \circ \widetilde{J}_h = i\, df$. Denote by $\mathcal{P}_h^j(\mathbf{v})$ the space of h-holomorphic homogeneous polynomials of degree j. This is the span of the monomials

$$X \mapsto \langle X, W\rangle_h^j, \quad W \in \mathbf{v}.$$

Set $r = |h|$ and $d\mu_r = e^{-(r/2)|v|^2}\,dv$, where dv is Lebesgue measure on $\mathbf{v}$. Let $\mathcal{F}_h(\mathbf{v})$ be the space of all h-holomorphic functions in $L^2(\mathbf{v}, \mu_r)$. Then $\mathcal{F}_h(\mathbf{v})$ is the Hilbert direct sum of the mutually orthogonal subspaces $\mathcal{P}_h^j(\mathbf{v})$.

We define a unitary representation π_h of N on $\mathcal{F}_h$ by the multiplier action

$$\pi_h(\exp(Y+Z))f(X) = \mu_h(Y,X)e^{i\langle h,Z\rangle}f(X+Y), \tag{3.9}$$

for $Y \in \mathbf{v}$, $Z \in \mathbf{z}$, where the multiplier is

$$\mu_h(Y,X) = \exp\left\{-\tfrac{1}{4}\langle Y + 2X, Y\rangle_h\right\}.$$

This representation is irreducible and is uniquely determined up to unitary equivalence by its central character $\chi_h(\exp Z) = e^{i\langle h,Z\rangle}$ (cf. [Fol89, Section 1.6]). Note that when $\dim \mathbf{z} > 1$ then "Planck's constant" h is a vector.

This realization of the representation of N with central character χ_h is particularly convenient for determining the action of the group MA on $\widehat{N}$. We will write $g \cdot X = \mathrm{Ad}(g)X$ for $X \in \mathbf{n}$ and $g \in MA$. The splitting in (3.2) is invariant under the adjoint action of MA. If $m \in M$ and $h \in \mathbf{z}$, then

$$m \cdot J_h X = \mathrm{Ad}(m)[h, \theta X] = [m \cdot h, \theta(m \cdot X)] = J_{m\cdot h}(m \cdot X)$$

for $X \in \mathbf{v}$, while the action of A commutes with J_h. Hence

$$\langle m \cdot X, m \cdot Y\rangle_{m\cdot h} = \langle X, Y\rangle_h, \qquad \langle a \cdot X, a \cdot Y\rangle_h = \langle X, Y\rangle_{a\cdot h},$$

for $X, Y \in \mathbf{v}$, $a \in A$ and $m \in M$. It follows that the Hermitian forms on $\mathbf{v}$ transform under $g \in MA$ by

$$\langle g \cdot X, g \cdot Y\rangle_{\theta(g)\cdot h} = \langle X, Y\rangle_h. \tag{3.10}$$

Let $f \in \mathcal{F}_h(\mathbf{v})$. Set

$$T(g)f(X) = a^{-p\beta/2} f(g^{-1} \cdot X)$$

for $g = ma \in MA$. Then we see from (3.9) and (3.10) that $T(g) : \mathcal{F}_h(\mathbf{v}) \to \mathcal{F}_{\theta(g)\cdot h}(\mathbf{v})$ is a unitary map and that

$$T(g)\pi_h(n) = \pi_{\theta(g)\cdot h}(gng^{-1})T(g) \tag{3.11}$$

for $g \in MA$ and $n \in N$ (cf. [KR83, Lemma 2.1]). Obviously we have $T(g) \circ T(g') = T(gg')$ for $g, g' \in MA$.

We can now describe the unitary dual of MN in terms of the unitary dual $\widehat{N}$ by the Mackey machine, as follows. For $h \in \mathbf{n}$, let M_h be the stabilizer of h in M and denote by T_h the restriction of the map T to M_h. By (3.11) we can extend π_h, for $0 \neq h \in \mathbf{z}$, from N to a representation σ_h of $M_h N$ by

$$\sigma_h(mn) = T_h(m)\pi_h(n). \tag{3.12}$$

For $h \in \mathbf{v}$ there are also the one-dimensional representations π_h of N given by

$$\pi_h(\exp X) = e^{i\langle h,X\rangle}$$

for $X \in \mathbf{n}$. We extend these representations to $M_h N$ by $\sigma_h(mn) = \pi_h(n)$.

Let $h \neq 0$ be either in $\mathbf{v}$ or in $\mathbf{z}$. Taking any irreducible representation τ of M_h on a finite-dimensional Hilbert space $\mathcal{E}_\tau$, we define a representation $\rho_{h,\tau}$ of $M_h N$ by

$$\rho_{h,\tau}(mn) = \overline{\tau}(m) \otimes \sigma_h(mn) \quad \text{for } m \in M_h\,,\ n \in N\,,$$

where $\overline{\tau}$ is the dual representation to τ. Set

$$\pi_{h,\tau} = \operatorname{Ind}_{M_h N}^{MN}(\rho_{h,\tau})\,.$$

Here we take the representation space to be all L^2 functions $f : M \to \mathcal{H}(\rho_{h,\tau})$ such that

$$f(km) = \rho_{h,\tau}(k) f(m) \quad \text{for } m \in M\,,\ k \in M_h\,,$$

with inner product

$$\langle f, g\rangle = \int_{M_h \backslash M} \langle f(x), g(x)\rangle \, d\overline{x}\,,$$

where $d\overline{x}$ denotes the invariant measure on $M_h \setminus M$. In this realization the action of MN is given by

$$\pi_{h,\tau}(nm) f(x) = I \otimes \pi_h(xnx^{-1}) f(xm) \tag{3.13}$$

for $m, x \in M$, $n \in N$. The representations $\pi_{h,\tau}$ are irreducible, and every infinite-dimensional irreducible representation of MN is obtained by this construction. The representations $\pi_{h,\tau}$ and $\pi_{h',\tau'}$ are equivalent iff $\tau \simeq \tau'$ and $|h| = |h'|$ ($h = h'$ if $\mathsf{F} = \mathbf{C}$ and $h \in \mathbf{z}$) (cf. [KLW77], [Cyg81]).

EXAMPLE 3.1 (Spherical representations). The representation $\pi = \pi_{h,\tau}$ is *spherical* if it contains an M-fixed vector ξ_π. By Frobenius reciprocity, this occurs iff the representation σ_h of M_h contains the trivial representation.

Case 1: $h \in \mathbf{z}$. Then σ_h contains the trivial representation iff τ occurs in T_h. Now $T_h = \sum \tau_j$, where τ_j is the restriction of T_h to the space of homogeneous h-holomorphic polynomials of degree j. But τ_j is irreducible. Indeed, by Kostant's double transitivity theorem (cf. [Wal73, §8.11.3]) M_h acts transitively on the unit sphere in $\mathbf{v}$ and hence the complexification of M_h acts transitively on $\mathbf{v} \setminus \{0\}$. Thus $\pi = \pi_{h,\tau}$ is spherical iff $\tau = \tau_j$ for some j. We will denote this representation by $\pi_{h,j}$, for $j = 0, 1, 2 \ldots$ Up to equivalence it only depends on the pair $(|h|, j)$ (resp. (h, j), in case $\mathsf{F} = \mathbf{C}$).

Let $\{e_i \mid i = 1, \ldots, d_j\}$ be an orthonormal basis for for $\mathcal{P}_h^j(\mathbf{v})$ and denote by $\overline{e}_i$ the linear functional $f \mapsto \langle f, e_i\rangle$. In the realization of π above

$$\xi_\pi(m) = d_j^{-1/2} \sum \overline{e}_i \otimes e_i \quad \text{for all } m \in M\,.$$

Case 2: $h \in \mathbf{v}$. Then π is spherical iff τ is trivial. Denote this representation by $\pi_{h,0}$. Up to equivalence it only depends on $|h|$. In the realization of π above $\xi_\pi(m) = 1$ for all $m \in M$.

Given $f, g \in \mathcal{H}(\pi_{h,\tau})$ we define the matrix entry function

$$\psi_{f,g}(x) = \langle \pi_{h,\tau}(x) f, g \rangle \tag{3.14}$$

for $x \in NM$. Using the explicit formula for the inner product on $\mathcal{H}(\pi_{h,\tau})$, we can express these functions as integrals over $M_h \setminus M$ of matrix entries of π_h. When $0 \neq h \in \mathbf{z}$, we obtain

$$\psi_{f,g}(nm) = \int_{M_h \setminus M} e^{i\langle h, x\cdot Z\rangle} \langle I \otimes \pi_h(\exp x \cdot Y) f(xm), g(x) \rangle \, d\bar{x} \tag{3.15}$$

for $m \in M$ and $n = \exp(Y + Z)$, with $Y \in \mathbf{v}$, $Z \in \mathbf{z}$. When $h \in \mathbf{v}$, the integral formula becomes

$$\psi_{f,g}(nm) = \int_{M_h \setminus M} e^{i\langle h, x\cdot Y\rangle} \langle f(xm), g(x) \rangle \, d\bar{x}. \tag{3.16}$$

EXAMPLE 3.2 (Spherical functions). When π is a spherical representation and $f = g = \xi_\pi$ is a normalized M-fixed vector, we denote the matrix entry by ψ_π. All the bounded spherical functions for the pair (MN, M) are of this form. From the explicit formulas for ξ_π in Example 3.1 we have the following integral representations of ψ_π.

Case 1: For the spherical representations $\pi = \pi_{h,j}$ with $0 \neq h \in \mathbf{z}$, let $E_j(h)$ be the orthogonal projection onto $\mathcal{P}^j_h(\mathbf{v})$. Then

$$\langle \pi(n)\xi_\pi(x), \xi_\pi(x) \rangle = d_j^{-1} \sum_i \langle \pi_h(xnx^{-1}) e_i, e_i \rangle = d_j^{-1} \operatorname{tr}(E_j(h)\pi_h(xnx^{-1})).$$

Hence we have the integral representation

$$\psi_\pi(n) = d_j^{-1} \int_{M_h \setminus M} e^{i\langle h, x\cdot Z\rangle} \operatorname{tr}(E_j(h)\pi_h(\exp x \cdot Y)) \, d\bar{x} \tag{3.17}$$

for $n = \exp(Y + Z)$, with $Y \in \mathbf{v}$, $Z \in \mathbf{z}$.

In case $\mathsf{F} = \mathbf{C}$, one has $M_h = M$ and no integration over M is needed in (3.17). In fact, this is true for the cases $\mathsf{F} = \mathbf{H}$ or $\mathbf{O}$ also. To see this, rewrite the integral as a double integral over the stabilizer subgroup M_Y and the quotient M/M_Y. Let $S_2(r)$ be the sphere of radius r in $\mathbf{z}$ with normalized invariant measure $d\zeta$. Since $M_Y \cdot h = S_2(|h|)$ by the double transitivity theorem, (3.17) factors as

$$\psi_\pi(n) = \left\{ d_j^{-1} \int_M \operatorname{tr}(E_j(h)\pi_h(\exp x \cdot Y)) \, dx \right\} \left\{ \int_{S_2(|h|)} e^{i\langle \zeta, Z\rangle} \, d\zeta \right\}.$$

But by the double transitivity theorem again, $M \cdot Y = M_h \cdot Y$, so we may replace integration over M by integration over M_h. In the integrand we have

$$\operatorname{tr}(E_j(h)\pi_h(\exp x\cdot Y)) = \operatorname{tr}(E_j(h)T(x)\pi_h(\exp Y)T(x^{-1})) = \operatorname{tr}(E_j(h)\pi_h(\exp Y))$$

for $x \in M_h$. Hence we conclude that

$$\psi_\pi(\exp(Y+Z)) = d_j^{-1}\operatorname{tr}(E_j(h)\pi_h(\exp Y))\Big\{\int_{S_2(|h|)} e^{i\langle\zeta,Z\rangle}\,d\zeta\Big\} \tag{3.18}$$

for all three cases $\mathsf{F} = \mathsf{C}, \mathsf{H}$ and O.

Case 2: For the spherical representations $\pi = \pi_{h,0}$ with $0 \neq h \in \mathbf{v}$ we have

$$\psi_\pi(n) = \int_{S_1(|h|)} e^{i\langle\eta,Y\rangle}\,d\eta, \tag{3.19}$$

where $S_1(r)$ is the sphere of radius r in $\mathbf{v}$ with normalized invariant measure $d\eta$.

These functions are calculated in terms of special functions (Bessel functions and Laguerre polynomials) in [Kor82] using the differential equations satisfied by spherical functions (cf. [BJR89, §8] for the representation-theoretic approach).

4. Decay of matrix entries. In this section we obtain decay estimates for the smooth matrix entries of infinite-dimensional irreducible unitary representations of MN. These estimates are more precise quantitative versions, in this special case, of the general results of Howe and Moore ([HM79]) for irreducible unitary representations of algebraic groups.

THEOREM 4.1. *Assume $q = \dim \mathbf{z} > 0$. Define $\gamma = 0$ if $q = 1$, and $\gamma = 3q - 3 + \frac{1}{2}\dim M$ if $q > 1$. Fix $0 \neq h \in \mathbf{z}$. Then for every integer $r > \gamma$ there is a constant C such that*

$$|\psi_{f,g}(\exp(Y+Z))| \leq C(1+|Y|)^{-s}(1+|Z|)^{-(q-1)/2}\|f\|_r\|g\|_r \tag{4.1}$$

for all $f, g \in \mathcal{H}^\infty(\pi_{h,\tau})$ and $Y \in \mathbf{v}$, $Z \in \mathbf{z}$, where $s = \lfloor r - \gamma \rfloor$.

Remark. Estimate (4.1) holds for all $f, g \in \mathcal{H}^r(\pi_{h,\tau})$, by the density of the C^∞ vectors.

We begin by establishing decay estimates for the matrix entries of π_h.

LEMMA 4.2. *Let $0 \neq h \in \mathbf{z}$, and let r be a positive integer. Then for $u, w \in \mathcal{H}^\infty(\pi_h)$ one has*

$$|h|^r|Y|^r|\langle\pi_h(\exp Y)u, w\rangle| \leq 2^r\|u\|_r\|w\|_r, \qquad Y \in \mathbf{v}, \tag{4.2}$$

where $\|u\|_r$ is the norm on $\mathcal{H}^r(\pi_h)$.

Proof. Suppose $Y, W \in \mathfrak{v}$. Then

$$\begin{aligned}\langle \pi_h(\exp Y)\pi_h(W)u, w\rangle &= \langle \pi_h(e^{\operatorname{ad} Y}W)\pi_h(\exp Y)u, w\rangle \\ &= \langle \pi_h(W + [Y, W])\pi_h(\exp Y)u, w\rangle .\end{aligned}$$

Rearranging terms, we obtain the identity

$$\begin{aligned}(4.3) \qquad i\langle h, [Y, W]\rangle\langle \pi_h(\exp Y)u, w\rangle &= \langle \pi_h(\exp Y)\pi_h(W)u, w\rangle \\ &\quad + \langle \pi_h(\exp Y)u, \pi_h(W)w\rangle .\end{aligned}$$

We may iterate this identity as follows. For non-negative integers j, k set

$$F(j, k) = \langle \pi_h(\exp Y)\pi_h(W)^j u, \pi_h(W)^k w\rangle$$

and write $Q = i\langle h, [Y, W]\rangle$. Then (4.3) implies that

$$Q \cdot F(j, k) = F(j + 1, k) + F(j, k + 1) .$$

Hence by induction we have

$$Q^r \cdot F(0, 0) = \sum_{j=0}^{r} \binom{r}{j} F(j, r - j) .$$

Since $|F(j, r - j)| \leq |W|^r \|u\|_r \|w\|_{r-j}$, we thus obtain the estimate

$$(4.4) \qquad |\langle h, [Y, W]\rangle|^r |\langle \pi_h(\exp Y)u, w\rangle| \leq |W|^r \sum_{j=0}^{r} \binom{r}{j} \|u\|_j \|w\|_{r-j} .$$

Taking $W = J_h Y$, we have $|W| = |h|\,|Y|$ and $\langle h, [Y, W]\rangle = \langle J_h Y, J_h Y\rangle = |h|^2 |Y|^2$. Also $\|u\|_j \leq \|u\|_r$ if $j \leq r$. Hence (4.2) follows from estimate (4.4). ∎

In case $q = 1$ one has $M_h = M$ and Theorem 4.1 follows from Lemma 4.2. Now suppose $q > 1$. Let $f \in \mathcal{H}^\infty(\pi_{h,\tau})$. Let R be the right regular representation of M on $L^2(M; \mathcal{H}(\pi_h))$. Since the restriction of $\pi_{h,\tau}$ to M is a subrepresentation of R, Lemma 2.1 implies that $f \in C^\infty(M; \mathcal{H}(\pi_h))$. Define

$$\Phi_f(n, m) = (\pi_{h,\tau}(n)f)(m)$$

for $n \in N$ and $m \in M$. We know that, for fixed n, Φ_f is a C^∞ function of m with values in $\mathcal{H}(\pi_h)$; we now show that it is jointly differentiable in n, m.

LEMMA 4.3. *The map Φ_f from $N \times M$ to $\mathcal{H}(\pi_h)$ is continuously differentiable, with differential at (n, m) the map*

$$(4.5) \qquad X, Y \mapsto \Phi_{\pi_{h,\tau}(\operatorname{Ad}(n^{-1})X+Y)f}(n, m)$$

for $Y \in \mathfrak{n}$, $X \in \mathfrak{m}$. (Here the tangent space to $N \times M$ at (n, m) is identified with $\mathfrak{n} + \mathfrak{m}$ via left translation.) Furthermore, $f(m) \in \mathcal{H}^1(\pi_h)$, and if

$s > 1 + \frac{1}{2}\dim M$ *then there is a constant* C *so that*

$$\sup_{m \in M} \|f(m)\|_1 \leq C\|f\|_s \,. \tag{4.6}$$

Remark. The norm on the left in (4.6) is on $\mathcal{H}^1(\pi_h)$ and the norm on the right is on $\mathcal{H}^s(\pi_{h,\tau})$.

Proof. We first observe that Φ_f is continuous from $N \times M$ to $\mathcal{H}(\pi_h)$. Indeed, since $\Phi_f(n,m) = \pi_h(mnm^{-1})f(m)$, this follows from the strong continuity of the representation π_h and the continuity of the function $m \mapsto f(m)$ given by Lemma 2.1.

Consider the partial differential of Φ_f relative to N. Since $f \in \mathcal{H}^\infty(\pi_{h,\tau})$, the map $Y \mapsto \pi_{h,\tau}(n \exp Y)f$ is strongly differentiable from $\mathbf{n}$ to $\mathcal{H}(\pi_{h,\tau})$. Together with Lemma 2.1 this implies that for fixed $m \in M$, the map $Y \mapsto \pi_{h,\tau}(n \exp Y)f(m)$ is strongly differentiable from $\mathbf{n}$ to $\mathcal{H}(\pi_h)$. Since

$$\pi_{h,\tau}(n \exp Y)f(m) = \pi_h(mnm^{-1})\pi_h(\exp m \cdot Y)f(m)\,,$$

it follows that $f(m) \in \mathcal{H}^1(\pi_h)$ and

$$\pi_h(m \cdot Y)f(m) = (\pi_{h,\tau}(Y)f)(m)\,.$$

Since $s - 1 > \frac{1}{2}\dim M$, the Sobolev inequality also gives the estimate

$$\|\pi_h(m \cdot Y)f(m)\| \leq C\|\pi_{h,\tau}(Y)f\|_{s-1} \leq C(1 + |Y|)\|f\|_s\,.$$

Replacing Y by $m^{-1} \cdot Y$ and taking the supremum over $Y \in \mathbf{n}$, $|Y| = 1$, we obtain (4.6). Furthermore, for fixed n, m we have

$$\left(\frac{d}{dt}\right)_{t=0} \Phi_f(n \exp tY, m) = \pi_{h,\tau}(n)(\pi_{h,\tau}(Y)f)(m) = \Phi_{\pi_{h,\tau}(Y)f}(n,m)\,.$$

The map $X \mapsto \pi_{h,\tau}(n \exp X)f$ is strongly differentiable from $\mathbf{m}$ to $\mathcal{H}(\pi_{h,\tau})$. Since

$$\Phi_f(n, m \exp tX) = \pi_{h,\tau}(n)\pi_{h,\tau}(\exp t\,\mathrm{Ad}(n)^{-1}X)f(m)\,,$$

Lemma 2.1 shows that the map $X \mapsto \Phi_f(n, m \exp X)$ is strongly differentiable from $\mathbf{m}$ to $\mathcal{H}(\pi_h)$ for each fixed n, m . Its differential is

$$X \mapsto \pi_{h,\tau}(n)\pi_{h,\tau}(\mathrm{Ad}(n)^{-1}X)f(m)\,.$$

Thus the partial maps $n \mapsto \Phi_f(n,m)$ and $m \mapsto \Phi_f(n,m)$ are differentiable. The partial differentials are continuous $\mathcal{H}(\pi_h)$-valued functions of n, m by the observations at the beginning of the proof. Hence Φ_f is a jointly C^1 function. The formula for its differential follows from the calculations just made. ∎

From Lemma 4.3 we next obtain the following pointwise estimates of Sobolev type for the $\mathbf{n}$ action on the components of vectors that are smooth relative to the joint action of M and N.

LEMMA 4.4. *Let r and k be positive integers such that $r > k + \frac{1}{2}\dim M$. Then there is a constant C so that*

$$\sup_{m\in M} \|f(m)\|_k \le C\|f\|_r. \tag{4.7}$$

Proof. Let $T \in U_{k-1}(\mathbf{n})$. Since $r - k + 1 > 1 + \frac{1}{2}\dim M$, we see from (4.6) that

$$\sup_{m\in M} \|\pi_{h,\tau}(T)f(m)\|_1 \le C\|\pi_{h,\tau}(T)f\|_{r-k+1} \le C'\|f\|_r$$

for some constants C, C' independent of f. But

$$\pi_{h,\tau}(T)f(m) = \pi_h(\mathrm{Ad}(m)T)f(m)$$

and M preserves the norm on $U_{k-1}(\mathbf{n})$. Hence

$$\|f(m)\|_k \le C\max_T \|\pi_{h,\tau}(T)f(m)\|_1$$

for some constant C independent of m, where T runs over a basis for $U_{k-1}(\mathbf{n})$. Combining these two estimates yields the lemma. ∎

We now turn to the problem of estimating the integrand in formula (3.15) for the matrix entry $\psi_{f,g}$. Let $f, g \in \mathcal{H}^\infty(\pi_{h,\tau})$. Define the function

$$\phi(Y, m) = \langle \pi_{h,\tau}(\exp Y)f(m), g(m)\rangle$$

for $Y \in \mathbf{v}$. By Lemma 4.3 we know that $\phi \in C^\infty(\mathbf{v} \times M)$. We have the following estimate for its M derivatives.

LEMMA 4.5. *Let $j, r, s \in \mathbb{N}$ with $r > s + j + \frac{1}{2}\dim M$. Let $D \in U_j(\mathbf{m})$. Then there is a constant C independent of f, g and Y so that*

$$\sup_{m\in M} |R(D)\phi(Y, m)| \le C(1 + |Y|)^{-s+2j}\|f\|_r\|g\|_r. \tag{4.8}$$

Proof. Let $X \in \mathbf{m}$ and $Y \in \mathbf{v}$. From the calculations in Lemma 4.3 we have

$$\begin{aligned} R(X)\phi(Y,m) = {} & \langle \pi_{h,\tau}(\exp Y)\pi_{h,\tau}(e^{-\operatorname{ad} Y}X)f(m), g(m)\rangle \\ & + \langle \pi_{h,\tau}(\exp Y)f(m), R(X)g(m)\rangle. \end{aligned}$$

But $e^{-\operatorname{ad} Y}X = X + [X,Y] + \frac{1}{2}[[X,Y],Y]$, so

$$\begin{aligned} R(X)\phi(Y,m) = {} & \langle \pi_h(\exp m\cdot Y)\pi_h(m\cdot([X,Y] + \tfrac{1}{2}[[X,Y],Y]))f(m), g(m)\rangle \\ & + \langle \pi_h(\exp m\cdot Y)R(X)f(m), g(m)\rangle \\ & + \langle \pi_h(\exp m\cdot Y)f(m), R(X)g(m)\rangle. \end{aligned}$$

Let $s \in \mathbb{N}$. Applying Lemma 4.2, we get the estimate

$$\begin{aligned} |R(X)\phi(Y,m)| \le {} & C(1 + |Y|)^{-s+2}\|f(m)\|_s\|g(m)\|_s \\ & + C(1 + |Y|)^{-s}\|R(X)f(m)\|_s\|g(m)\|_s \\ & + C(1 + |Y|)^{-s}\|f(m)\|_s\|R(X)g(m)\|_s, \end{aligned}$$

where the constant C does not depend on f, g or Y. Hence if $r > s + 1 + \frac{1}{2}\dim M$ then Lemma 4.4 furnishes the bound

$$|R(X)\phi(Y,m)| \leq C(1+|Y|)^{-s+2}\|f\|_r\|g\|_r,$$

where the constant C does not depend on f, g or Y. This proves the lemma when $j = 1$. Now iterate this calculation to obtain (4.8) for elements of order j in $U(\mathfrak{m})$. ∎

Completion of proof of Theorem 4.1. Recall that by formula (3.15) we have

$$\psi_{f,g}(\exp(Y+Z)) = \int_{M_h\backslash M} e^{i\langle h, m\cdot Z\rangle}\phi(Y,m)d\overline{m}. \tag{4.9}$$

By Kostant's Double Transivity Theorem (cf. [Wal73, §8.11.3]) the orbit $M \cdot h$ is the $(q-1)$-sphere of radius $|h|$ in $\mathfrak{v}$. The integral (3.15) is thus the Fourier transform of a C^∞ density on a $(q-1)$-dimensional sphere. Estimate (4.1) follows by [Hor83, Theorems 7.7.6 and 7.7.14] and Lemma 4.5 with $j = q - 1$. ∎

Now we consider the representations of MN induced from characters of N. In this case the rate of decay of smooth matrix entries is determined by the dimension of the unit sphere in N/Z. (As a special case one obtains the well-known decay of the Bessel functions J_ν.)

THEOREM 4.6. *Assume that* $p = \dim\mathfrak{v} > 1$. *Let* $0 \neq h \in \mathfrak{v}$. *If* $r > p - 1 + \frac{1}{2}\dim M$ *then there is a constant* C *such that*

$$|\psi_{f,g}(\exp Y)| \leq C(1+|Y|)^{-(p-1)/2}\|f\|_r\|g\|_r \tag{4.10}$$

for all $Y \in \mathfrak{v}$ *and* $f, g \in \mathcal{H}^r(\pi_{h,\tau})$.

Proof. Recall from (3.16) that $\psi_{f,g}$ has the integral representation

$$\psi_{f,g}(\exp Y) = \int_{M_h\backslash M} e^{i\langle h, m\cdot Y\rangle}\langle f(m), g(m)\rangle\, d\overline{m}. \tag{4.11}$$

By Lemma 2.1, f and g are in $C^{p-1}(M, \mathcal{E}_\tau)$, with norms in C^{p-1} bounded by the $\mathcal{H}^r(\pi_{h,\tau})$ norm. By Kostant's Double Transivity Theorem (cf. [Wal73, §8.11.3]) the orbit $M \cdot h$ is the $(p-1)$-sphere of radius $|h|$ in $\mathfrak{v}$. The integral (4.11) is thus the Euclidean Fourier transform of a C^{p-1} density on a $(p-1)$-dimensional sphere. Estimate (4.10) follows from [Hor83, Theorems 7.7.6 and 7.7.14] (cf. [Ste87] for further references). ∎

5. Operator Fourier transforms.

5. Operator Fourier transforms. Let π be an infinite-dimensional irreducible unitary representation of MN. If $f \in L^1(N)$ then the integral

$$\int_N f(x)\pi(x)\,dx$$

converges absolutely and defines a bounded operator $\pi(f)$. We now use the asymptotics of the smooth matrix entries for π to show that if f is locally integrable on N and has some decay at infinity, this integral still converges absolutely in a suitable weak operator topology. This gives a "generalized Fourier transform" $\pi(f)$ which maps $\mathcal{H}^r(\pi)$ continuously to $\mathcal{H}^{-r}(\pi)$ for some $r \geq 0$.

THEOREM 5.1. *Let $f \in L^1_{\mathrm{loc}}(N)$, and assume that f satisfies a bound*

$$|f(x)| \leq C(1+\mathcal{N}(x))^{-Q+a} \tag{5.1}$$

for x outside some bounded set, where $a < p+q-1$. Let $\pi = \pi_{h,\tau}$, with $0 \neq h \in \mathbf{z}$, and let $\xi, \eta \in \mathcal{H}^\infty(\pi)$. Then the integral

$$\int_N \langle \pi(x)\xi, \eta\rangle f(x)\,dx \tag{5.2}$$

is absolutely convergent. Denote its value by $\langle \pi(f)\xi, \eta\rangle$. Then $\pi(f)$ extends to a continuous linear operator from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, where $r = \lfloor p + 3q + \frac{1}{2}\dim M\rfloor$. It satisfies the intertwining properties

$$\pi(L(n')R(n)f) = \pi(n')\pi(f)\pi(n^{-1}), \tag{5.3}$$

$$\pi(f^m) = \pi(m^{-1})\pi(f)\pi(m), \tag{5.4}$$

for $m \in M$, $n, n' \in N$, where $f^m(x) = f(m^{-1}xm)$. Furthermore, the adjoint operator $\pi(f)^* = \pi(f^*)$, where $f^*(n) = \overline{f(n^{-1})}$.

Proof. Since f is locally integrable, we may write $f = f_1 + f_2$, where $f_1 \in L^1(N)$ and f_2 satisfies (5.1) for all $x \in N$. The theorem is true for f_1, so it suffices to consider the case $f = f_2$.

Write $\psi(Y,Z) = \langle \pi(\exp(Y+Z))\xi, \eta\rangle$, where $Y \in \mathbf{v}$, $Z \in \mathbf{z}$. If we take $r = \lfloor p+3q+\frac{1}{2}\dim M\rfloor$ in Theorem 4.1, then the exponent $s \geq p$ in equation (4.1). Thus we obtain the estimate

$$|\psi(Y,Z)| \leq C'(1+|Y|)^{-p}(1+|Z|)^{-(q-1)/2}\|\xi\|_r\|\eta\|_r,$$

where the constant C' only depends on h, τ. Since we are assuming that (5.1) holds for all x, we can thus bound (5.2) by $CC'\|\xi\|_r\|\eta\|_r$ times the integral

$$\int_{\mathbf{v}}\int_{\mathbf{z}} (1+|Y|^2)^{-p/2}(1+|Z|)^{-(q-1)/2}(1+|Y|^2+|Z|)^{-(Q-a)/2}\,dZ\,dY.$$

So it suffices to show that this integral converges. Replacing $|Z|$ by $(1+|Y|^2)^{-1}|Z|$ in the middle factor in the integrand and combining terms, we can majorize the integral by

$$\int_{\mathbf{v}}\int_{\mathbf{z}} (1+|Y|^2)^{-(2p+2q-a)/2}(1+(1+|Y|^2)^{-1}|Z|)^{-(p+3q-a-1)/2}\,dZ\,dY.$$

After the change of variable $Z \mapsto (1+|Y|^2)Z$ this integral becomes

$$\int_{\mathbf{v}} \int_{\mathbf{z}} (1+|Y|^2)^{-(p+b)/2}(1+|Z|)^{-q-c}\,dZ\,dY\,, \tag{5.5}$$

where $b = p - a$ and $c = (p+q-1-a)/2$. If $a < p+q-1$ then $b > 0$ and $c > 0$, and hence (5.5) converges. The intertwining properties follow immediately from the definition of $\pi(f)$. ∎

Now we consider the case of representations induced from characters of N.

THEOREM 5.2. *Let $f \in L^1_{\mathrm{loc}}(N)$, and assume that f satisfies* (5.1) *for x outside some bounded set, where $a < (p-1)/2$. Let $\pi = \pi_{h,\tau}$ with $0 \neq h \in \mathbf{v}$, and let $\xi, \eta \in \mathcal{H}^\infty(\pi)$. Then the integral* (5.2) *is absolutely convergent. Denote its value by $\langle \pi(f)\xi, \eta\rangle$. Then $\pi(f)$ extends to a continuous linear operator from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, where $r = \lfloor p + \frac{1}{2}\dim M \rfloor$. It satisfies the intertwining properties* (5.3), (5.4) *and $\pi(f)^* = \pi(f^*)$.*

Proof. We apply the same argument, *mutatis mutandis*, as in Theorem 5.1, using the decay estimate from Theorem 4.6. This leads to the majorizing integral (5.5) with $b = p - 1 - 2a$ and $c = (p-a)/2$. If $a < (p-1)/2$ then $b > 0$ and $c > 0$, and hence (5.5) converges. ∎

EXAMPLE 5.1 (Biradial functions). Following Korányi [Kor82], we call a function f on N *biradial* if $f^m = f$ for all $m \in M$. Assume f is biradial and satisfies (5.1) for values of a as in Theorems 5.1 or 5.2. By (5.4) the operator $\pi(f)$ commutes with $\pi|_M$, and hence it can be studied in terms of its action in the finite-dimensional M-isotypic subspaces.

Suppose $\pi = \pi_{h,\tau}$. From (3.13) we calculate that $\pi(f)$ is multiplication by the operator $I_\tau \otimes \pi_h(f)$ if $h \in \mathbf{z}$, where I_τ is the identity operator on $\overline{\mathcal{E}_\tau}$. If $h \in \mathbf{v}$ then $\pi(f) = \widehat{f}(h)I$ is a scalar multiple of the identity. Here $\widehat{f}$ is the Euclidean Fourier transform of f defined relative to the invariant form on $\mathbf{n}$.

EXAMPLE 5.2. Suppose that π is a spherical representation and $\xi_\pi \in \mathcal{H}(\pi)$ is a normalized M-fixed vector (cf. Example 3.1). Let f be biradial as in Example 5.1. Then

$$\pi(f)\xi_\pi = \langle f, \psi_\pi\rangle \xi_\pi\,, \tag{5.6}$$

where

$$\langle f, \psi_\pi\rangle = \int_N f(n)\psi_\pi(n)\,dn \tag{5.7}$$

($\psi_\pi(n)$ is the spherical function associated with π). The absolute convergence of (5.7) follows from the general decay estimates of the previous section. This also evident from the explicit formulas for ψ_π (cf. Example 3.2

and [Far82]) and the asymptotic behavior of Bessel functions. The map $f \mapsto \langle f, \psi_\pi \rangle$ is the *M-spherical transform* of f.

If $\pi = \pi_{h,j}$ with $h \in \mathbf{z}$, then we can also calculate the M-spherical transform of f as a partial trace of the operator $\pi_h(f)$. Indeed, from (3.17) and the M-invariance of f, (5.7) can be written as

$$\langle f, \psi_\pi \rangle = d_j^{-1} \operatorname{tr}(E_j(h)\pi_h(f)) . \tag{5.8}$$

For example, let $f(n) = \mathcal{N}(n)^{\lambda - Q}$ and $\pi = \pi_{h,j}$, with $h \in \mathbf{z}$. Then from (5.8) and [Cow82, Theorem 8.1] we have

$$\langle f, \psi_\pi \rangle = 2^{1-p/2}\pi^{(p+q+1)/2}|h|^{-\lambda/2} b_{h,j}(\lambda) , \tag{5.9}$$

where

$$b_{h,j} = \frac{\Gamma((p+4j+2-\lambda)/4)}{\Gamma((p+4j+2+\lambda)/4)} \frac{\Gamma(\lambda/2)}{\Gamma((Q-\lambda)/4)\Gamma((p+2-\lambda)/4)} .$$

(Note that Cowling's "kth block" corresponds to the subspace $\mathcal{P}_h^j(\mathbf{v})$ with $j = k - 1$.)

6. Homogeneous kernels. We now apply the results of the previous section to define the operator Fourier transform of certain non-integrable homogeneous densities on N. For ease of notation we identify A with the multiplicative group $(0, \infty)$ under $a_t \mapsto t$, and write $d^*t = t^{-1}\,dt$ for the invariant measure. We identify $\mathbf{a}_{\mathbf{c}}^*$ with $\mathbf{C}$ by $c\beta \mapsto c$. Then $a_t^\lambda = t^\lambda$ and $2\rho \mapsto Q$. We say that a function g on $N \setminus \{1\}$ is *homogeneous of weight* μ if $g(\delta_t n) = t^\mu g(n)$ for $n \neq 1$. We shall be particularly interested in the following class of homogeneous functions.

LEMMA 6.1. *Let $f \in \mathcal{S}(N)$. Define*

$$f_\lambda(n) = \int_A f(a \cdot n) a^{-\lambda + 2\rho}\, da . \tag{6.1}$$

For $n \in N \setminus \{1\}$ and $\Re(\lambda) < Q$ the integral is absolutely convergent. It defines a jointly C^∞ function of n, λ which is holomorphic in λ and homogeneous of weight $\lambda - Q$ in n. If $0 < \Re(\lambda) < Q$ then $f_\lambda \in L^1_{\mathrm{loc}}(N)$.

Proof. Let $Y \in \mathbf{v}$, $Z \in \mathbf{z}$. Then

$$f_\lambda(\exp(Y+Z)) = \int_0^\infty t^{-\lambda+2\rho} f(\exp(tY + t^2 Z))\, d^*t ,$$

where $d^*t = t^{-1}\,dt$. Since $f(\exp(Y+Z))$ is a Schwartz-class function of (Y, Z), the statements in the lemma follow by standard Mellin transform techniques. ∎

EXAMPLE 6.1. Suppose P is a homogeneous polynomial of weight $\mu \geq 0$ on $\mathbf{n}$. View P as a function on N via the exponential map, and set

$$f(n) = P(n)e^{-\mathcal{N}(n)^4}.$$

Then $f \in \mathcal{S}(N)$ and

$$f_\lambda(n) = 4\Gamma((Q+\mu-\lambda)/4)P(n)\mathcal{N}(n)^{\lambda-\mu-Q} \tag{6.2}$$

for $n \neq 1$.

Let π be an irreducible unitary representation of MN. We would like to define the (operator-valued) Fourier transform $\pi(f_\lambda)$ as a meromorphic function of $\lambda \in \mathbf{a}_{\mathbf{c}}^*$. We identify f_λ with the measure $f_\lambda(n)\,dn$ on MN and start with a range of λ values where this measure is locally finite. The decay of the matrix entries of π then compensates for the singularity of the measure at infinity.

THEOREM 6.2. *Let $f \in \mathcal{S}(N)$. Suppose $\pi = \pi_{h,\tau}$, with $h \in \mathbf{z}$ (resp. $h \in \mathbf{v}$), is an infinite-dimensional irreducible unitary representation of MN. Let $\xi, \eta \in \mathcal{H}^\infty(\pi)$. Then the integral*

$$\int_A \int_N \langle \pi(n)\xi, \eta\rangle f(a \cdot n) a^{-\lambda+2\rho}\, da\, dn \tag{6.3}$$

is absolutely convergent in the strip $0 < \Re(\lambda) < p+q-1$ (resp. $0 < \Re(\lambda) < (p-1)/2$), and is a holomorphic function of λ. Denote its value by $\langle \pi(f_\lambda)\xi, \eta\rangle$ for λ in this strip. Then $\pi(f_\lambda)$ extends to a continuous linear operator from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, where $r = \lfloor p + 3q + \frac{1}{2}\dim M \rfloor$ (resp. $r = \lfloor p + \frac{1}{2}\dim M\rfloor$).

Proof. Write $\psi(n) = \langle \pi(n)\xi, \eta\rangle$ and

$$F_\sigma(n) = \int_A |f(a\cdot n)| a^{-\sigma+2\rho}\, da,$$

where $\sigma = \Re(\lambda)$. Then F_σ is homogeneous of weight $\sigma - Q$ and smooth away from 1. Hence $F_\sigma(n) \leq M_\sigma(f)\mathcal{N}(n)^{-Q+\sigma}$, where

$$M_\sigma(f) = \sup_{\mathcal{N}(n)=1} F_\sigma(n). \tag{6.4}$$

If we replace the integrand in (6.3) by its absolute value, then we get the integral

$$\int_N F_\sigma(n)|\psi(n)|\, dn. \tag{6.5}$$

By Theorems 5.1 and 5.2 this integral is absolutely convergent, under the stated conditions on σ. Hence (6.3) is absolutely convergent by the Fubini–Tonelli theorem and the operator $\pi(f_\lambda)$ has the stated continuity properties.

The holomorphic dependence of the integral on λ follows from Morera's theorem, using the absolute convergence (cf. [Kat66, Ch. VII, §1.1] for general properties of holomorphic operator-valued functions). ∎

Remarks. 1. The function f_λ is locally integrable for λ in the strips occurring in Theorem 6.2. Since $n \mapsto \langle \pi(n)\xi, \eta\rangle f_\lambda(n)$ is in $L^1(N)$ for all $\xi, \eta \in \mathcal{H}^r(\pi)$, where $r = p$ (resp. $r = 2p$), it follows by the dominated convergence theorem that

$$\pi(f_\lambda) = \lim_{R\to\infty} \int_{\mathcal{N}(n)<R} f_\lambda(n)\, dn$$

in the weak operator topology on $\mathrm{Hom}(\mathcal{H}^r(\pi), \mathcal{H}^{-r}(\pi))$. In particular, $\pi(f_\lambda)$ is a well-defined operator-valued function of f_λ for each λ in the indicated range.

2. Let M_f be defined by (6.4). The proof of Theorem 6.2 furnishes the bounds

$$\|\pi(f_\lambda)\|_{r,-r} \le C_\sigma M_\sigma(f) \tag{6.6}$$

for the operator norm, with the constant C_σ uniformly bounded in compact subsets of the interval $(0, p+q-1)$ (resp. $(0, (p-1)/2)$). Here $\|A\|_{r,-r}$ denotes the operator norm on $\mathrm{Hom}(\mathcal{H}^r(\pi), \mathcal{H}^{-r}(\pi))$ and $r = \lfloor p+3q+\frac{1}{2}\dim M\rfloor$ (resp. $r = \lfloor p + \frac{1}{2}\dim M\rfloor$). Theorem 6.2 holds for any measurable function f on N for which $M_f < \infty$.

The estimates (6.6) just obtained for the operators $\pi(f_\lambda)$ only require that f be bounded and have a certain polynomial rate of decay at infinity. Now we use the smoothness of f to show that $\pi(f_\lambda)$ is actually bounded on $\mathcal{H}(\pi)$ for $\Re(\lambda) > 0$, and can be meromorphically continued in the parameter λ as an operator on $\mathcal{H}^\infty(\pi)$.

THEOREM 6.3. *Let π be an infinite-dimensional irreducible unitary representation of MN, and let $f \in \mathcal{S}(N)$. The regularized operator*

$$\pi(\widetilde{f}_\lambda) = \Gamma(\lambda)^{-1}\pi(f_\lambda),$$

initially defined for $\Re(\lambda) > 0$, extends to a continuous operator from $\mathcal{H}^\infty(\pi)$ to $\mathcal{H}^\infty(\pi)$ for all $\lambda \in \mathbf{C}$ with the following properties:

1. *For $\Re(\lambda) > 0$ it is bounded on $\mathcal{H}^s(\pi)$ for all $s \ge 0$.*
2. *For $\Re(\lambda) > -k$ and $s \ge 0$ it is a bounded operator from $\mathcal{H}^{s+k}(\pi)$ to $\mathcal{H}^s(\pi)$ and is a holomorphic function of λ in the strong operator topology.*
3. *$\pi(\widetilde{f}_\lambda)|_{\lambda=0} = (\int_N f(n)\, dn)I$.*

The key ingredient in the proof of Theorem 6.3 is the following.

LEMMA 6.4. *Fix orthonormal bases* $\{Y_i \mid i = 1, \ldots, p\}$ *for* v *and* $\{Z_j \mid j = 1, \ldots, q\}$ *for* z *and define the partial Laplacians*

$$\Delta_1 = \sum_{i=1}^{p} Y_i^2, \qquad \Delta_2 = \sum_{j=1}^{q} Z_j^2 \tag{6.7}$$

in $U(\mathfrak{n})$. *Suppose* $\pi = \pi_{h,\tau}$. *Then* $\pi(\Delta_i) = -|h|^2 I$, *where* $i = 1$ *if* $h \in \mathfrak{v}$ *and* $i = 2$ *if* $h \in \mathfrak{z}$.

Proof. Since M acts orthogonally on $\mathfrak{v}$ and $\mathfrak{z}$, the elements Δ_1 and Δ_2 are M-invariant. They are A-homogeneous of weights 2β and 4β, respectively.

Consider first the case $0 \neq h \in \mathfrak{z}$. In the representation π_h of N the operator Δ_2 acts by the M-invariant scalar $-|h|^2$. From the description of $\pi_{h,\tau}|_N$ in Section 3 we see that $\pi_{h,\tau}(\Delta_2) = -|h|^2$ also. When $0 \neq h \in \mathfrak{v}$, then $\pi_{h,\tau}(\Delta_2) = 0$ but $\pi_{h,\tau}(\Delta_1) = -|h|^2$. ∎

Proof of Theorem 6.3 (cf. [Sch71, Prop. 3.3]). For $t > 0$ define

$$T_f(t, \pi) = \int_N f(n)\pi(a_t \cdot n)\, dn. \tag{6.8}$$

Since $f \in \mathcal{S}(N)$ and N is cocompact in MN, it is clear that $T_f(t, \pi) : \mathcal{H}(\pi) \to \mathcal{H}^\infty(\pi)$ continuously. Let λ be in the strip in Theorem 6.2. Since the integral (6.3) is absolutely convergent in this range, we may invert the order of integration relative to A and N. After making the changes of variable $n \mapsto a_t^{-1} \cdot n$ followed by $t \mapsto t^{-1}$, we obtain $\pi(f_\lambda)$ as a Mellin transform:

$$\pi(f_\lambda) = \int_0^\infty t^\lambda T_f(t, \pi)\, d^*t, \tag{6.9}$$

where the integral converges absolutely in the $\mathcal{B}(\mathcal{H}^r(\pi), \mathcal{H}^{-r}(\pi))$ norm, with r as in Theorem 6.2. We now prove that the integral representation (6.9) of $\pi(f_\lambda)$ has much better convergence properties than the original defining integral.

The convergence and analytic continuation in λ of the integral (6.9) depends on the behavior of $T_f(t, \pi)$ as $t \to 0$ and $t \to \infty$. We consider first the case of large t. We view functions on N as right M-invariant functions on MN, and we denote the left regular representation of MN and of $U(\mathfrak{m} + \mathfrak{n})$ by L. We will be considering L on various function spaces that will be clear from the context.

Let $X \in \mathfrak{n}$. Then an easy calculation shows that

$$\pi(X)T_f(t, \pi) = T_{L(a_t^{-1} \cdot X)f}(t, \pi).$$

Hence this equation holds for all $X \in U(\mathfrak{n})$. Let $\pi = \pi_{h,\tau}$. Set $c = |h|^{-2}$ and $D = \Delta_i$, where $i = 1$ if $h \in \mathfrak{v}$ and $i = 2$ if $h \in \mathfrak{z}$. Then D is A-homogeneous, and by Lemma 6.4 we have

$$T_f(t,\pi) = -ct^{-\mu} T_{L(D)f}(t,\pi)\,, \tag{6.10}$$

where $\mu = 2\beta$ (resp. 4β) if $h \in \mathfrak{v}$ (resp. $h \in \mathfrak{z}$). Iterating (6.10) and using the unitarity of π we get the operator-norm estimate

$$\|T_f(t,\pi)\| \le c^n t^{-n\mu} \|L(D^n)f\|_{L^1} \tag{6.11}$$

for every positive integer n. It follows that the integral

$$B_f(\lambda,\pi) = \int_1^\infty t^\lambda T_f(t,\pi)\, d^*t$$

converges absolutely in the operator norm on $\mathcal{H}(\pi)$ for all $\lambda \in \mathbf{C}$. Clearly $B_f(\lambda,\pi)$ is a holomorphic function of λ which satisfies the bound

$$\|B_f(\lambda,\pi)\| \le \frac{c^n}{n\mu - \sigma} \|L(D^n)f\|_{L^1} \tag{6.12}$$

if $\Re(\lambda) = \sigma < n\mu$. When $X \in \mathfrak{m}$ then

$$\pi(X)T_f(t,\pi) = T_{L(X)f}(t,\pi) + T_f(t,\pi)\pi(X)\,.$$

Combining this with the action of D, we conclude that for all positive integers n, s, there is a constant $C_{n,s}$ so that

$$\|T_f(t,\pi)\xi\|_s \le C_{n,s} t^{-n\mu} \|f\|_{1,n\mu+s} \|\xi\|_s\,,$$

for all $\xi \in \mathcal{H}^\infty(\pi)$. Here the norm on f is from $L^1_{n\mu+s}(MN)$, the $n\mu+s$ times differentiable vectors for the left regular representation of MN on $L^1(MN)$. It follows that $B_f(\lambda,\pi) : \mathcal{H}^s(\pi) \to \mathcal{H}^s(\pi)$ continuously for all $\lambda \in \mathbf{C}$ and all $s \ge 0$.

Consider now the behavior of $T_f(t,\pi)$ as $t \to 0$. If $\xi \in \mathcal{H}^\infty(\pi)$ and $Y \in \mathfrak{n}$, then the function $t \mapsto \pi(\exp(a_t \cdot Y))\xi$ is infinitely differentiable from $\mathbf{R}$ to $\mathcal{H}$. Its kth derivative is of the form

$$\pi(\exp(a_t \cdot Y))\pi(Q_k(t,Y))\xi\,,$$

where $Q_k(t,Y) \in U_k(\mathfrak{n})$ is a polynomial in t and Y. It follows by dominated convergence that the function $t \mapsto T_f(t,\pi)\xi$ extends to an infinitely differentiable function from $\mathbf{R}$ to $\mathcal{H}$. Denoting its kth t-derivative by $T_f^{(k)}(t,\pi)\xi$, we have

$$T_f^{(k)}(t,\pi)\xi = \int_{\mathfrak{n}} f(\exp Y)\pi(\exp(a_t \cdot Y))\pi(Q_k(t,Y))\xi\, dY\,. \tag{6.13}$$

Let $X \in \mathfrak{m} + \mathfrak{n}$ and $Y \in \mathfrak{n}$. Then

$$\pi(X)\pi(\exp Y)\xi = \pi(\exp Y)\pi(e^{-\operatorname{ad} Y}X)\xi\,.$$

But $(\operatorname{ad} Y)^3 X = 0$, so $e^{-\operatorname{ad} Y} X$ is a quadratic polynomial in Y. Hence for any integer $s \geq 0$ we have the estimate

$$\|\pi(\exp Y)\xi\|_s \leq C_s(1+|Y|)^{2s}\|\xi\|_s,$$

where the constant C_s does not depend on Y. Using this estimate in (6.13), we obtain a bound

$$\|T_f^{(k)}(t,\pi)\xi\|_s \leq C_{k,s}\|\xi\|_{k+s} \int_{\mathfrak{n}} |f(\exp Y)|(1+|Y|)^{k+2s}\, dY \tag{6.14}$$

when $0 \leq t \leq 1$.

From (6.14) it follows that when $\Re(\lambda) > 0$, the integral

$$C_f(\lambda,\pi) = \int_0^1 t^{\lambda} T_f(t,\pi)\, d^*t$$

defines a bounded operator on $\mathcal{H}^s(\pi)$ for all $s \geq 0$. To carry out the analytic continuation, we integrate by parts as usual and obtain the operator identity

$$C_f(\lambda,\pi) = \sum_{j=0}^{k-1} \frac{(-1)^j}{\lambda(\lambda+1)\ldots(\lambda+j)}\pi^{(j)}(f) + \frac{(-1)^k}{\lambda(\lambda+1)\ldots(\lambda+k-1)} \int_0^1 t^{\lambda+k} T_f^{(k)}(t,\pi)\, d^*t$$

on $\mathcal{H}^\infty(\pi)$, where we have set $\pi^{(j)}(f) = T_f^{(j)}(1,\pi)$. With a regularizing gamma factor, this equation can be written as

$$\Gamma(\lambda)^{-1}C_f(\lambda,\pi) = \Gamma(\lambda+k)^{-1}\sum_{j=0}^{k-1}(-1)^j\Big(\prod_{i=j+1}^{k-1}(\lambda+i)\Big)\pi^{(j)}(f) + (-1)^k\Gamma(\lambda+k)^{-1}\int_0^1 t^{\lambda+k}T_f^{(k)}(t,\pi)\, d^*t. \tag{6.15}$$

From (6.14) and (6.15) we conclude that $\lambda \mapsto \Gamma(\lambda)^{-1}C_f(\lambda,\pi)$ has an analytic continuation to a continuous operator on $\mathcal{H}^\infty(\pi)$ which depends holomorphically on λ. More precisely, if $\Re(\lambda) > -k$ then for all $s \geq 0$

$$\Gamma(\lambda)^{-1}C_f(\lambda,\pi) : \mathcal{H}^{s+k}(\pi) \to \mathcal{H}^s(\pi)$$

is bounded, with operator norm bounded by

$$C_{k,s,\lambda}\int_{\mathfrak{n}} |f(\exp Y)|(1+|Y|)^{k+2s}\, dY.$$

Since $\pi(f_\lambda) = B_f(\lambda,\pi) + C_f(\lambda,\pi)$ on the strip $0 < \Re(\lambda) < p-1$, this gives the desired analytic continuation.

Finally, from (6.15) with $k = 1$ we have

$$\lim_{\lambda \to 0} \lambda \pi(f_\lambda) = T_f(1,\pi) - \int_0^1 T_f^{(1)}(t,\pi)\,dt = T_f(0,\pi).$$

But $\lim_{t\to 0} \pi(a_t \cdot n) = I$, so by dominated convergence

$$T_f(0,\pi) = \Big(\int_N f(n)\,dn \Big) I.$$

This completes the proof. ∎

7. Whittaker transform. We now study the Fourier transform of functions on the homogeneous space $N\backslash G$ relative to the group $A \times N$ (direct product). Recall that since A normalizes N, it acts by *left* translations on $N \backslash G$. For $\phi \in C_c^\infty(N \backslash G)$ and $\lambda \in \mathfrak{a}_{\mathbf{c}}^*$ set

$$\phi_\lambda(g) = \int_A \phi(ag) a^{-\lambda-\rho}\,da. \tag{7.1}$$

Then $\phi_\lambda \in C^\infty(N \backslash G)$ and $\phi_\lambda(ag) = a^{\lambda+\rho}\phi_\lambda(g)$ for $a \in A$.

THEOREM 7.1. *Suppose $\pi = \pi_{h,\tau}$, with $h \in \mathfrak{z}$ (resp. $h \in \mathfrak{v}$), is an infinite-dimensional irreducible unitary representation of MN. Let $\xi, \eta \in \mathcal{H}_\infty(\pi)$. Then for every $\phi \in C_c^\infty(N \backslash G)$ the integral*

$$\int_N \phi_\lambda(m^* n) \langle \pi(n)\xi, \eta\rangle\,dn \tag{7.2}$$

converges absolutely in the half-plane $\Re(\lambda) > -(p+q-1)/2$ (resp. $\Re(\lambda) > -(p-1)/4$), and is a holomorphic function of λ. Denote its value by $\langle W_\pi(\phi_\lambda)\xi, \eta\rangle$ for λ in this half-plane. Then $W_\pi(\phi_\lambda)$ extends to a continuous linear operator from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, where $r = \lfloor p + 3q + \frac{1}{2}\dim M \rfloor$ (resp. $r = \lfloor p + \frac{1}{2}\dim M \rfloor$).

Remark. When $\Re(\lambda) > 0$ then (7.2) is absolutely convergent for all $\xi, \eta \in \mathcal{H}(\pi)$, and $W_\pi(\phi_\lambda)$ is a bounded operator on $\mathcal{H}(\pi)$. An essential point of the theorem is that the integral (7.2) remains absolutely convergent for λ in an open half-plane containing the imaginary axis, provided ξ, η are sufficiently smooth vectors.

Proof. If $n = n(Y, Z)$, then by (3.5), $\phi_\lambda(m^* n) = t^{\lambda+\rho}\phi_\lambda(k)$ for some $k \in K$, where

$$t^{-2} = (1 + |Y|^2)^2 + |Z|^4 \geq \tfrac{1}{16}(1 + \mathcal{N}(n))^4.$$

Hence if $\sigma = \Re(\lambda) > -\rho$ then

$$|\phi_\lambda(nm^*)| \leq 4^{\rho+\sigma}\{\sup_{k\in K} |\phi_\lambda(k)|\}(1 + \mathcal{N}(n))^{-Q-2\sigma}. \tag{7.3}$$

The theorem now follows immediately from (7.3) and Theorems 5.1 and 5.2. ∎

We shall call $W_\pi(\phi_\lambda)$ the *Whittaker transform* of ϕ. The Whittaker transform obviously intertwines the right action of N on $C_c^\infty(N\backslash G)$ with right multiplication by $\pi(n)^{-1}$:

$$W_\pi(R(n)\phi_\lambda) = W_\pi(\phi_\lambda)\pi(n)^{-1}\,. \tag{7.4}$$

Since π is also a representation of M, the Whittaker transforms of left and right translates of ϕ by elements of M are related by

$$W_\pi(R(m)\phi_\lambda)\pi(m) = \pi(m)W_\pi(L(m')\phi_\lambda)\,, \tag{7.5}$$

where $m' = m^*m^{-1}m^{*-1}$. This follows by the change of variable $n \mapsto mnm^{-1}$ in the defining integral, together with the property $L(m)\phi_\lambda = (L(m)\phi)_\lambda$ for $m \in M$.

EXAMPLE 7.1. Suppose $\phi \in C_c^\infty(MN \setminus G)$, i.e. ϕ is invariant under left translations by both N and M. Then by (7.5)

$$W_\pi(R(m)\phi_\lambda) = \pi(m)W_\pi(\phi_\lambda)\pi(m^{-1})\,.$$

Thus the Whittaker transform intertwines right translation by $m \in M$ on $MN \setminus G$ with conjugation by the operator $\pi(m)$. In particular, if ϕ is also right M-invariant, then $W_\pi(\phi_\lambda)$ commutes with $\pi(m)$, $m \in M$, and hence preserves the M-isotypic components of π.

Let π be a *spherical* representation of MN with M-fixed vector ξ_π and spherical function ψ_π. The distribution $w_\pi(\lambda)$ on $MN \setminus G$ defined by

$$\langle \phi, w_\pi(\lambda)\rangle = \langle W_\pi(\phi_\lambda)\xi_\pi, \xi_\pi\rangle \tag{7.6}$$

is right M-invariant, and depends holomorphically on λ. If ϕ is also right M-invariant, then

$$\langle W_\pi(R(n)\phi_\lambda)\xi_\pi, \xi_\pi\rangle = \langle W_\pi(\phi_\lambda)\pi(n^{-1})\xi_\pi, \xi_\pi\rangle = \psi_\pi(n^{-1})\langle W_\pi(\phi_\lambda)\xi_\pi, \xi_\pi\rangle\,,$$

since $W_\pi(\phi_\lambda)^*\xi_\pi$ is proportional to ξ_π. Distributions on $MN \setminus G$ with these trasformation properties under right M and N translations are called *M-spherical Whittaker vectors* (cf. [Far82, §II.1]). When $\Re(\lambda)$ satisfies the conditions in Theorem 7.1 we can represent $w_\pi(\lambda)$ by the integral

$$\langle \phi, w_\pi(\lambda)\rangle = \int_N \phi_\lambda(m^*n)\psi_\pi(n)\,dn$$

for $\phi \in C_c^\infty(MN \setminus G)$.

Now we consider the analytic continuation of the Whittaker transform in the parameter λ and its continuity properties as an operator on the space $\mathcal{H}^\infty(\pi)$. For the M-spherical Whittaker distributions in Example 7.1 this was done by Faraut (*loc. cit.*), modifying techniques of Schiffmann. Our method is similar.

THEOREM 7.2. *Let $\phi \in C_c^\infty(N \setminus G)$. For all $s \geq 0$, the operator-valued function $\lambda \mapsto W_\pi(\phi_\lambda)$ extends to an entire analytic function from $\mathfrak{a}_\mathbb{C}^*$ to* $\mathrm{End}(\mathcal{H}^s(\pi))$.

Proof. We essentially follow [Sch71, Théorème 3.1], replacing the center of $U(\mathfrak{n})$ by suitable M-invariants in $U(\mathfrak{n})$ and noting that Schiffmann's estimates also imply continuity of $W_\pi(\phi_\lambda)$ on the spaces $\mathcal{H}^s(\pi)$. The argument goes as follows.

Set $\phi^a(g) = \phi(a^{-1}ga)$ and $\pi^a(n) = \pi(a^{-1}na)$ for $a \in A$. Assuming $\Re(\lambda) > 0$, we use the absolute convergence of the integrals to justify the following calculation:

$$\begin{aligned} W_\pi(\phi_\lambda) &= \int_N \int_A \phi(am^*n)a^{-\lambda-\rho}\pi(n)\,dn\,da \\ &= 2\int_N \int_A \phi(a^{-2}m^*n)a^{2\lambda+2\rho}\pi(n)\,dn\,da \\ &= 2\int_N \int_A \phi^a(m^*n)a^{2\lambda}\pi^a(n)\,dn\,da\,, \end{aligned}$$

where in the second line we made the change of variable $a \mapsto a^{-2}$ and in the third line $n \mapsto a^{-1}na$.

Take $\Omega = \mathrm{supp}(\phi)$. Then by [Sch71, Lemma 3.3] there exists $T < \infty$ and a compact set $C \subset N$ so that $a_t^{-1}m^*na_t \in N\Omega$ implies that $t \leq T$ and $n \in C$. Thus if we define

$$F_\phi(a,\pi) = \int_N \phi^a(m^*n)\pi^a(n)\,dn\,, \tag{7.7}$$

then the integrand is zero outside C. Hence $F_\phi(a,\pi)$ is a bounded operator on $\mathcal{H}(\pi)$ with

$$\|F_\phi(a,\pi)\| \leq \|\phi\|_\infty \mathrm{meas}(C) \tag{7.8}$$

bounded uniformly in a. By Fubini's theorem we can write the Whittaker transform in terms of F_ϕ as

$$W_\pi(\phi_\lambda) = 2\int_0^T F_\phi(a_t,\pi)t^{2\lambda}\,d^*t\,. \tag{7.9}$$

(From the bound (7.8) we know *a priori* that this integral converges in the strong operator topology on $\mathcal{H}(\pi)$ when $\Re(\lambda) > 0$.)

The convergence and analytic continuation of (7.9) for $\Re(\lambda) \leq 0$ depends on the behavior of $F_\phi(a_t,\pi)$ as $t \to 0$. The invariance of the measure dn yields the relation

$$F_\phi(a,\pi)\pi^a(X) = \int_N (R(X^{\mathrm{T}})\phi^a)(m^*n)\pi^a(n)\,dn$$

for $X \in U(\mathbf{n})$. (Here $X \mapsto X^{\mathrm{T}}$ is the canonical anti-automorphism on $U(\mathbf{n})$.) In particular, taking $X = D^k$ as in the proof of Theorem 6.3, we obtain the identity

$$F_\phi(a,\pi) = c^k a^{k\mu} \int_N (R(D^k)\phi^a)(m^*n)\pi^a(n)\,dn \tag{7.10}$$

for all positive integers k, where μ is a positive multiple of β. We now recall the key estimate.

LEMMA 7.3 (Schiffmann). *Let $\phi \in C^\infty(N \setminus G)$. Then for every compact set $\Gamma \subset G$, $T \in \mathbf{R}$, and $u \in U(\mathbf{g})$,*

$$\sup_{g\in\Gamma, t\le T} |(R(u)\phi^{a_t})(g)| < \infty\,. \tag{7.11}$$

Proof. See [Sch71, Lemme 3.4]. ∎

Completion of proof of Theorem 7.2. Let $u \in U_s(\mathbf{m}+\mathbf{n})$ be A-homogeneous of weight $\nu \geq 0$. By the nilpotence of $\mathrm{ad}(\mathbf{n})$ there are elements $u_j \in U_s(\mathbf{m}+\mathbf{n})$ homogeneous of weight $\nu + j\beta$, a positive integer J, and polynomials p_j on N so that

$$\mathrm{Ad}(n^{-1})u = \sum_{j=0}^{J} p_j(n)u_j\,.$$

Hence

$$\pi(u)\pi^a(n) = a^\nu \pi^a(n)\pi^a(\mathrm{Ad}(n^{-1})u) = \pi^a(n)\sum_{j=0}^{J} a^{-j\beta}p_j(n)\pi(u_j)\,.$$

It follows from (7.10) that

$$\begin{aligned}\pi(u)F_\phi(a,\pi) & \\ &= c^k \sum_{j=0}^{J} a^{k\mu - j\beta}\Big(\int_N p_j(n)(R(D^k)\phi^a)(m^*n)\pi^a(n)\,dn\Big)\pi(u_j)\,.\end{aligned} \tag{7.12}$$

Given u and a positive integer r, we take k so that $k\mu \geq (r+J)\beta$. Recalling that the function $n \mapsto (L(D^k)\phi^a)(nm^*)$ has compact support uniformly in a, we obtain from (7.11) and (7.12) the estimate

$$\sup_{t\le T} \|\pi(u)F_\phi(a_t,\pi)\xi\| \leq Ct^r\|\xi\|_s \tag{7.13}$$

for $\xi \in \mathcal{H}^\infty(\pi)$, where C is a constant depending on k, u and ϕ. The proof of the theorem now follows from (7.9) and (7.13) by the same Mellin transform techniques as in Theorem 6.3. ∎

8. Functional equations. Let $\phi \in C_c^\infty(N\setminus G)$. For $\lambda \in \mathbf{a}_{\mathbf{c}}^*$ and $n \in N$, set $\widetilde{\phi}_\lambda(n) = \phi_\lambda(m^*n)$.

THEOREM 8.1. *Let $f \in \mathcal{S}(N)$ and $\Re(\lambda) > 0$. Then the integral*

$$f_\mu * \widetilde{\phi}_\lambda(g) = \int_N f_\mu(y)\widetilde{\phi}_\lambda(y^{-1}g)\,dy \tag{8.1}$$

converges absolutely when $0 < \Re(\mu) \leq Q$ and $g \in G$.

Suppose $\pi = \pi_{h,\tau}$, with $h \in \mathfrak{z}$ (resp. $h \in \mathfrak{v}$), is an infinite-dimensional irreducible unitary representation of MN. Let $\xi, \eta \in \mathcal{H}^\infty(\pi)$. Then the integral

$$\int_N \langle \pi(x)\xi, \eta\rangle f_\mu * \widetilde{\phi}_\lambda(x)\,dx \tag{8.2}$$

is absolutely convergent in the strip $0 < \Re(\mu) < p+q-1$ (resp. $0 < \Re(\mu) < (p-1)/2$). Denote its value by $\langle \pi(f_\mu * \widetilde{\phi}_\lambda)\xi, \eta\rangle$ for μ in this strip. Then $\pi(f_\mu * \widetilde{\phi}_\lambda)$ extends to a continuous linear operator from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, where $r = \lfloor p + 3q + \frac{1}{2}\dim M \rfloor$ (resp. $r = \lfloor p + \frac{1}{2}\dim M \rfloor$).

COROLLARY 8.2. *If $\Re(\lambda) > 0$ then*

$$\pi(f_\mu)W_\pi(\phi_\lambda) = \pi(f_\mu * \widetilde{\phi}_\lambda) \tag{8.3}$$

as operators from $\mathcal{H}^r(\pi)$ to $\mathcal{H}^{-r}(\pi)$, for μ, r in the indicated range.

Before proving the theorem we consider the convolution of two particular functions of the homogeneous norm. Define

$$\Phi(x) = \int_N \mathcal{N}(y)^{-Q+a}(1 + \mathcal{N}(y^{-1}x))^{-Q-b}\,dy. \tag{8.4}$$

LEMMA 8.3. *If $0 < a \leq Q$ and $b > 0$, then there is a constant $C = C(a,b)$ so that*

$$\Phi(x) \leq C(1 + \mathcal{N}(x))^{-Q+a}$$

for all $x \in N$.

Proof. We estimate Φ first on the set $\{\mathcal{N}(x) \leq 1\}$. By the triangle inequality (3.4), $|\mathcal{N}(x) - \mathcal{N}(y)| \leq \mathcal{N}(y^{-1}x)$. Hence if $\mathcal{N}(x) \leq 1$ and $\mathcal{N}(y) \geq 2$ then $1 + \mathcal{N}(y^{-1}x) \geq \mathcal{N}(y)$. So splitting the integral for Φ over the sets $\{\mathcal{N}(y) \leq 2\}$ and $\{\mathcal{N}(y) > 2\}$, we obtain the bound

$$\Phi(x) \leq \int_{\mathcal{N}(y)\leq 2} \mathcal{N}(y)^{-Q+a}\,dy + \int_{\mathcal{N}(y)>2} \mathcal{N}(y)^{-Q-b}\,dy.$$

Each of these integrals is finite, hence Φ is bounded on $\{\mathcal{N}(x) \leq 1\}$.

Now assume $\mathcal{N}(x) > 1$. On the set

$$D = \{y \in N \mid \mathcal{N}(y^{-1}x) \leq \tfrac{1}{2}\mathcal{N}(x)\}$$

one has $\mathcal{N}(y) \geq \frac{1}{2}\mathcal{N}(x)$ by the triangle inequality. It follows that

$$\int_D \mathcal{N}(y^{-1}x)^{-Q+a}(1+\mathcal{N}(y))^{-Q-b}\,dy$$
$$\leq 2^{Q+b}(1+\mathcal{N}(x))^{-Q-b}\int_D \mathcal{N}(xy^{-1})^{-Q+a}\,dy$$
$$\leq C(1+\mathcal{N}(x))^{-Q-b}\mathcal{N}(x)^a\,,$$

where the last inequality follows from the integral formulas for powers of the function $\mathcal{N}$ [KS71]. The integral over the complement of D can be majorized by

$$2^{Q-a}\mathcal{N}(x)^{-Q+a}\int_N (1+\mathcal{N}(y))^{-Q-b}\,dy$$

if $0 < a \leq Q$, with the last integral finite since $b > 0$. Combining these two estimates, we obtain a bound

$$\Phi(x) \leq C\max\{(1+\mathcal{N}(x))^{-Q-b}\mathcal{N}(x)^a, \mathcal{N}(x)^{-Q+a}\}$$

when $\mathcal{N}(x) > 1$, which is equivalent to the bound in the lemma. ∎

Proof of Theorem 8.1. Take $g = xg'$ with $x \in N$ in (8.1). By (6.4) and (7.3) we can bound the integrand by

$$C\mathcal{N}(xy^{-1})^{-Q+\tau}(1+\mathcal{N}(y))^{-Q-2\sigma}\,,$$

where $\sigma = \Re(\lambda)$, $\tau = \Re(\mu)$, and C does not depend on x. Now apply Lemma 8.3 and Theorems 5.1 and 5.2 to see that (8.1) converges absolutely and that the operator $\pi(f_\mu * \widetilde{\phi}_\lambda)$ exists as stated. ∎

Proof of Corollary 8.2. Let $\xi, \eta \in \mathcal{H}^\infty(\pi)$. Then $W_\pi(\phi_\lambda)\xi \in \mathcal{H}^\infty(\pi)$ by Theorem 7.1 and $\pi(f_\mu)W_\pi(\phi_\lambda)\xi \in \mathcal{H}^{-r}(\pi)$ by Theorem 6.2. Furthermore, we have the absolutely convergent integral representations

$$\langle\pi(f_\mu)W_\pi(\phi_\lambda)\xi,\eta\rangle = \int_N f_\mu(y)\langle\pi(y)W_\pi(\phi_\lambda)\xi,\eta\rangle\,dy$$
$$= \int_N\int_N f_\mu(y)\widetilde{\phi}_\lambda(x)\langle\pi(yx)\xi,\eta\rangle\,dy\,dx\,.$$

But by Fubini's theorem and Theorem 8.1 this double integral is

$$\langle\pi(f_\mu * \widetilde{\phi}_\lambda)\xi,\eta\rangle$$

for λ, μ in the indicated range. ∎

We illustrate the results of this paper in the context of the spherical principal series representations of G, previously treated by Faraut (*loc. cit.*). Recall that the (unnormalized) Kunze–Stein intertwining operator A_λ acts on $\phi \in C_c^\infty(MN \setminus G)$ by the integral

$$A_\lambda(\phi_\lambda)(g) = \int_N \phi_\lambda(m^* ng)\, dn \tag{8.5}$$

when $\Re(\lambda) > 0$. One has

$$A_\lambda(\phi_\lambda)(mang) = a^{-\lambda+\rho} A_\lambda(\phi_\lambda)(g),$$

for $m \in M$, $a \in A$, $n \in N$, so that A_λ intertwines the spherical principal series representations of G with parameters λ and $-\lambda$.

By the Bruhat factorization and a change of variable, (8.5) can be expressed as

$$A_\lambda(\phi_\lambda)(m^* g) = \int_N \mathcal{N}(n)^{2\lambda - Q} \phi_\lambda(m^* ng)\, dn \tag{8.6}$$

([KS71], [Sch71], cf. [Goo76, Ch. IV, Sec. 1.3]). Throughout the following let

$$f(n) = e^{-\mathcal{N}(n)^4} = e^{-|Y|^4 - |Z|^2}$$

if $n = \exp(2Y + Z)$. Then $f \in \mathcal{S}(N)$ and $f_\lambda(n) = 4\Gamma((Q-\lambda)/4)\mathcal{N}(n)^{\lambda - Q}$ (cf. Example 6.1). Hence

$$A_\lambda(\phi_\lambda)(m^* g) = 4^{-1}\Gamma((Q-2\lambda)/4)^{-1} f_{2\lambda} * \widetilde{\phi}_\lambda(g)$$

when $\Re(\lambda)$ is small and positive, as in Theorem 8.1.

Take the function $\phi \in C_c^\infty(MN \setminus G)$ so that $\phi_\lambda = P_\lambda$, the *Poisson kernel*, characterized by the property $P_\lambda(k) = 1$ for all $k \in K$. Since P_λ is the unique K-fixed vector, one has

$$A_\lambda(P_\lambda)(g) = c(\lambda) P_{-\lambda}(g),$$

where $c(\lambda)$ is a scalar (the Harish-Chandra *c-function*). We recall that $c(\lambda)$ can be evaluating by setting $g = 1$ and using the Iwasawa decomposition (3.5):

$$c(\lambda) = \int_{\mathfrak{v}} \int_{\mathfrak{z}} ((1 + |Y|^2)^2 + |Z|^2)^{-(2\lambda + Q)/4}\, dY\, dZ\,.$$

Calculating this integral in polar coordinates, one obtains the well-known result [Hel70, Ch. III, Cor. 1.17]

$$c(\lambda) = c_0 2^{-\lambda}\Gamma(\lambda) \Big/ \left(\Gamma\left(\frac{2\lambda + Q}{4}\right)\Gamma\left(\frac{2\lambda + p + 2}{4}\right)\right) \tag{8.7}$$

with a constant c_0 depending only on p, q. In particular, $c(\lambda)$ is a meromorphic function.

Set $\gamma(\lambda) = 4c(\lambda)\Gamma((Q-2\lambda)/4)$ and define the *normalized* intertwining operator

$$\mathcal{A}_\lambda(\phi_\lambda) = \gamma(\lambda)^{-1} f_{2\lambda} * \widetilde{\phi}_\lambda \tag{8.8}$$

for $\Re(\lambda) > 0$. Then $\mathcal{A}_\lambda(P_\lambda) = P_{-\lambda}$.

EXAMPLE 8.1 (Whittaker transform of Poisson kernel). Let π be an infinite-dimensional irreducible representation of MN. From (8.8) and Corollary 8.2 we have

$$\gamma(\lambda)^{-1}\pi(f_{2\lambda})\pi(\widetilde{P}_\lambda) = \pi(\widetilde{P}_{-\lambda}) \tag{8.9}$$

for $\Re(\lambda) > 0$ and sufficiently small. Here each side of the equation is a continuous operator from $\mathcal{H}^\infty(\pi)$ to $\mathcal{H}^{-\infty}(\pi)$ (cf. Theorems 6.2, 7.2), and has a meromorphic continuation in the parameter λ. Hence (8.9) holds for all but a discrete set of λ (cf. [Sch71, Théorème 3.3]).

EXAMPLE 8.2 (Spherical Whittaker transform). Consider a spherical representation π of MN with M-fixed vector ξ_π. From Example 5.2 we have

$$\gamma(\lambda)^{-1}\pi(f_{2\lambda})\xi_\pi = \gamma_\pi(\lambda)\xi_\pi\,, \tag{8.10}$$

where $\gamma_\pi(\lambda)$ is a meromorphic function of λ. Take $\phi \in C_c^\infty(MN \setminus G/M)$. Then by Example 7.1 and Corollary 8.2 we have

$$\gamma(\lambda)^{-1}\pi(f_{2\lambda} * \widetilde{\phi}_\lambda)\xi_\pi = \gamma_\pi(\lambda)\pi(\widetilde{\phi}_\lambda)\xi_\pi \tag{8.11}$$

for $\Re(\lambda) > 0$ and sufficiently small. But each side of this equation has a meromorphic continuation in the parameter λ (cf. Theorem 7.2), and hence (8.11) holds for all λ. We may write this equation in terms of the M-spherical Whittaker distribution $w_\pi(\lambda)$ and the normalized intertwining operator $\mathcal{A}_\lambda$ as

$$\langle \mathcal{A}_\lambda\phi_\lambda, w_\pi(-\lambda)\rangle = \gamma_\pi(\lambda)\langle \phi_\lambda, w_\pi(\lambda)\rangle\,. \tag{8.12}$$

When $\pi = \pi_{h,j}$, with $h \in \mathbf{z}$, one obtains an explicit formula for the multiplier $\gamma_\pi(\lambda)$ using (5.9):

$$\begin{aligned}\gamma_\pi(\lambda) &\\ &= |h|^{-\lambda}\frac{\Gamma((p+4j+2-2\lambda)/4)}{\Gamma((p+4j+2+2\lambda)/4)}\frac{\Gamma((p+2+2\lambda)/4)}{\Gamma((p+2-2\lambda)/4)}\frac{\Gamma((Q+2\lambda)/4)}{\Gamma((Q-2\lambda)/4)}\end{aligned} \tag{8.13}$$

(cf. [Far82, Prop. II.4], where our variables $\lambda, |h|, j$ correspond to Faraut's $s, \lambda/4, n$ respectively).

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