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A PURE ARITHMETICAL DEFINITION OF THE CLASS GROUP

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Let K be an algebraic number field. We denote by R_K its ring of integers, by $H(K)$ the class group, and by $h(K)$ the class number.

It is well known that arithmetic properties of R_K depend on the structure of $H(K)$. It is a classical result that the condition $h(K) = 1$ is both necessary and sufficient for a unique factorization in R_K. Further, if $h(K) \neq 1$, then the length of factorization of an arbitrary integer of K does not depend on the factorization if and only if $h(K) = 2$ (see [1]). Śliwa [4] gives similar characterizations for other class numbers. In [2] one can find arithmetic characterization for fields with class groups either cyclic or being p-groups of a particular form.

The aim of this note is to describe the structure of class group in a pure arithmetical way. This solves Problem 32 from [3].

The following definition will be useful in this connection:

An algebraic integer d from R_K is said to be *completely irreducible* if it is irreducible and d^n has a unique factorization for every natural n. The following lemma gives the main properties of such integers:

LEMMA. (i) *An algebraic integer d is completely irreducible if and only if there exists a prime ideal $\mathfrak{p}$ such that*

$$dR_K = \mathfrak{p}^{\operatorname{ord}[\mathfrak{p}]},$$

where $[\mathfrak{p}]$ denotes the class from $H(K)$ to which $\mathfrak{p}$ belongs.

(ii) *There exists a natural number m (depending on d) such that for every completely irreducible d' the length of any factorization of dd' is less than or equal to m. Let* $\operatorname{ord} d$ *be the smallest such m. Then*

$$\operatorname{ord} d = \max(2, \operatorname{ord}[\mathfrak{p}]), \quad \textit{where } \mathfrak{p} | dR_K.$$

Proof. (i) is proved in [2].

(ii) Let d' be completely irreducible and let

$$dd' = a_1 \dots a_r$$

For $t\in(-\infty, 1)$ we put $f_1(t)=\varphi(t)$. If $t\in\langle 1, \infty)$, then $t=s_n^v$ for some real v and natural n. Let us put $f_1(s_n^v)=x_n^v$.

Similarly, for $t\in(-1, \infty)$ we put $f_2(t)=\varphi(t)$ and $f_2(t_n^u)=y_n^u$ for $t_n^u\in(-\infty, -1\rangle$.

One can check that the function $F=(f_1, f_2)$ is the one looked for.

Conversely, assume now that $F(R)=R^2$ and $D_1\cup D_2=R$, where $D_i=\{t\in R: f_i'(t) \text{ exists}\}$ $(i=1, 2)$. The functions f_1, f_2 satisfy the Banach condition (T_2) on the sets D_1, D_2, respectively (see [2], Chap. VII, Theorem 10.1, p. 234, and Chap. IX, p. 279(¹)). Hence the sets

$$N_i=\{y\in f_i(D_i): |f_i^{-1}(\{y\})\cap D_i|>\aleph_0\}, \quad i=1, 2,$$

have Lebesgue measure zero. Therefore, the sets $M_i=R-N_i$ $(i=1, 2)$ are of power $\mathfrak{c}$. Let $S_i=F(D_i)\cap(M_1\times M_2)$, $i=1, 2$. The sets S_1, S_2, M_1, M_2 satisfy the conditions of Sierpiński's theorem, and hence the proof of the theorem is complete.

We shall see further that a function $F=(f_1, f_2)$ defined in Theorem 1 does not exist when we assume that at least one of the coordinate functions f_1 or f_2 is Lebesgue measurable. This will follow from Theorem 3. First we prove the following

THEOREM 2. *Let $F=(f_1, f_2)$, where $f_1\in T_2(R)$ and f_2 is an arbitrary function. Let $F(R)$ be a Lebesgue measurable subset of R^2. Then $\lambda_2(F(R))=0$, where λ_2 is the Lebesgue measure on the plane R^2.*

Proof. There exist two disjoint sets A and B such that $A\cup B=R$, $\lambda(B)=0$, and $|f_1^{-1}(\{y\})|\leqslant\aleph_0$ for each $y\in A$. According to Fubini's theorem we can write

$$\begin{aligned}\lambda_2(F(R)) &= \lambda_2(F(R)\cap(A\times R))\\ &= \int_A \lambda(F(R)\cap(\{x\}\times R))\,d\lambda(x) = \int_A \lambda\big(f_2(f_1^{-1}(\{x\}))\big)\,d\lambda(x)=0\end{aligned}$$

because $|f_2(f_1^{-1}(\{x\}))|\leqslant\aleph_0$ for each $x\in A$.

COROLLARY. *Let $f_1\in VBG(R)$ and let f_2 be continuous on R. Assume that $F=(f_1, f_2)$. Then $\lambda_2(F(R))=0$.*

Proof. Let $\{E_n: n=1, 2, \ldots\}$ be a family of sets such that

$$\bigcup_{n=1}^{\infty} E_n=R \quad \text{and} \quad f_1\in VB(E_n) \quad \text{for each } n.$$

Let us consider any fixed set E_n. The function $f\restriction E_n$ can be extended to a function $g_n\in VB(R)$ ([2], Chap. VII, Lemma 4.1, p. 221). Let $F_n=(g_n, f_2)$. Since F_n is a Borel function, the set $F_n(R)$ is analytic ([1], Chap. III, Section 38, Proposition 5, p. 457). Therefore, the set $F_n(R)$ is Lebesgue measur-

(¹) In [2] this fact is shown for intervals, but it is true for any subset of R.

able ([1], Chap. III, Section 39, p. 482). Hence, by Theorem 2, we have $\lambda_2(F_n(R)) = 0$, which implies $\lambda_2(F_n(E_n)) = \lambda_2(F(E_n)) = 0$. Finally, $\lambda_2(F(R)) = 0$.

THEOREM 3. *Let f_1: $R \to R$ and f_2: $R \to R$. Assume that*

(i) *the function f_1 is Lebesgue measurable*;

(ii) *for each $x \in R$ there exists at least one of the derivatives $f_1'(x), f_2'(x)$*;

(iii) *$F(R)$ is a Lebesgue measurable subset of R^2, where $F = (f_1, f_2)$.*

Then $\lambda_2(F(R)) = 0$.

Proof. Let us put $D_i = \{t \in R\colon f_i'(t) \text{ exists}\}$, $i = 1, 2$. There exists a sequence $\{K_n\}_{n=1}^{\infty}$ of closed subsets of R such that $\lambda(R - K_n) < 1/n$ and $f_1 \restriction K_n$ is continuous for $n = 1, 2, \ldots$ Let us consider the set $D_2 \cap K_n$ for a certain fixed n. The function f_2 is differentiable on $D_2 \cap K_n$, and so $f_2 \in VBG(D_2 \cap K_n)$ ([2], Chap. VII, Theorem 10.1, p. 234). Let $\{A_j\colon j = 1, 2, \ldots\}$ be a family of sets such that

$$D_2 \cap K_n = \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad f_2 \in VB(A_j) \quad \text{for} \quad j = 1, 2, \ldots$$

For every j there exists an extension of $f_2 \restriction A_j$ to a function $g_j \in VB(R)$. Of course, there also exists an extension of $f_1 \restriction K_n$ to a continuous function h on R. For the vector function $H = (h, g_j)$ we have $\lambda_2(H(R)) = 0$ (see the Corollary), whence $\lambda_2(H(A_j)) = 0$. This implies $\lambda_2(F(D_2 \cap K_n)) = 0$ and, consequently,

$$\lambda_2(F(D_2 \cap \bigcup_{n=1}^{\infty} K_n)) = 0.$$

The function f_2 is differentiable on the set $D_2 - \bigcup_{n=1}^{\infty} K_n$ and $\lambda(D_2 - \bigcup_{n=1}^{\infty} K_n) = 0$, whence

$$\lambda(f_2(D_2 - \bigcup_{n=1}^{\infty} K_n)) = 0$$

([2], Chap. VII, Theorem 6.5, p. 227). Consequently, we obtain

$$\lambda_2(F(D_2 - \bigcup_{n=1}^{\infty} K_n)) = 0$$

and, finally, $\lambda_2(F(D_2)) = 0$. Thus $F(D_1)$ is Lebesgue measurable. Let us put $\varphi(t) = f_1(t)$ for $t \in D_1$ and $\varphi(t) = 0$ for $t \in R - D_1$. The function φ satisfies the Banach condition (T_2) on R. Let $G = (\varphi, f_2)$. The sets $G(D_1) = F(D_1)$ and $G(R - D_1)$ are Lebesgue measurable, and from Theorem 2 we obtain $\lambda_2(G(R)) = 0$. Hence $\lambda_2(F(D_1)) = 0$. Finally, $\lambda_2(F(R)) = 0$.

We formulate now other versions of Theorems 2 and 3, omitting the assumption of Lebesgue measurability of $F(R)$. To prove these theorems we should apply the same methods as those used in the proofs of Theorems 2 and 3.

THEOREM 2'. *Let* $F = (f_1, f_2)$, *where* $f_1 \in T_2(R)$ *and* f_2 *is an arbitrary function. Then* $\lambda_2^i(F(R)) = 0$, *where* λ_2^i *denotes the inner Lebesgue measure on the plane* R^2.

THEOREM 3'. *Let* $f_1\colon R \to R$, $f_2\colon R \to R$, *and* $F = (f_1, f_2)$. *Assume that the function* f_1 *is Lebesgue measurable and that for each* $x \in R$ *there exists at least one of the derivatives* $f_1'(x)$, $f_2'(x)$. *Then* $\lambda_2^i(F(R)) = 0$.

Finally, we pose the following problem:

PROBLEM (**P 1276**). Does there exist a function $F = (f_1, f_2)\colon I \to I \times I$, where $I = \langle 0, 1\rangle$, such that $F(I) = I \times I$ and for each $x \in I$ there exists at least one of the derivatives $f_1'(x)$, $f_2'(x)$ (as in Theorem 1 one can prove that from the existence of F CH would follow)?

Let us mention that if we put above an open or half-open interval (instead of I) as the domain of F, then the existence of F is, as in Theorem 1, equivalent to CH.

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