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HOMEOTOPY GROUPS OF 3-MANIFOLDS – AN ISOMORPHISM THEOREM

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1. Introduction. In 1963, McCarty [11] (Theorem 4.21) proved that if M is a compact manifold with boundary ($\dim M > 1$) and F is a finite subset of the interior of M, then the group of homeomorphisms (compact-open topology) of $M-F$ is topologically isomorphic to the group of homeomorphisms of M that leave F fixed. Sprows [15] showed that if M is a compact closed 2-manifold, $\hat{M}$ is the 2-manifold obtained from M by removing the interiors of a finite number of disjoint discs in M, and F is the set of centers of these discs, then the homeotopy group of M (see Section 2) is isomorphic to that of $M-F$. Tondra [16] extended Sprows' result to the non-compact case, i.e., M is connected, separable, metrizable and without boundary. The purpose of this note is to extend these results to manifolds of dimension three and to replace discs by handlebodies and the centers of discs by standard cores of the handlebodies.

2. Definitions and notation. Let M denote a connected, separable, metrizable 3-manifold with (or without) boundary. Let $V_1, V_2, \ldots, V_n$ be disjoint orientable handlebodies in M°, the interior of M, i.e., V_i is homeomorphic to a regular neighborhood in E^3 of a point or of the union of (more than one) polygonal arcs $ax_1b, ax_2b, \ldots, ax_kb$, where no two of the arcs intersect except at a and b. In fact, suppose that V_i is a regular neighborhood in M of the graph $C_i = \bigcup ax_jb$ or the point C_i. Denote $\bigcup C_i$ by F, and the manifold $M - \bigcup V_i^\circ$ by $\hat{M}$. The set C_i is the core of V_i.

Let $G(M)$ denote the group of all homeomorphisms of M onto itself topologized by the compact-open topology, and let $H(M)$ denote $G(M)$ modulo its identity component. The group $H(M)$ is the *homeotopy group* of M. Note that $G(M)$ is locally path connected (see [2] and [4]) so that the identity component consists of those homeomorphisms that are isotopic to the identity, denoted by id_M. If N is a compact subset of M, let $G(M; N)$

denote the group of homeomorphisms of M that leave N pointwise fixed and let $G(M \operatorname{rel} N)$ denote the group of homeomorphisms of M that take N onto N. Then $H(M; N)$ and $H(M \operatorname{rel} N)$ denote $G(M; N)$ and $G(M \operatorname{rel} N)$ modulo their identity components.

If φ: $M \times I \to M$ is an isotopy, denote by φ_τ the homeomorphism of M that takes each point x to $\varphi(x, \tau)$. I shall frequently refer to the isotopy φ as to the "isotopy $\{\varphi_\tau\}$".

In the present context, not all homeomorphisms of $\hat{M}$ extend to M nor do all those that do extend to M extend to homeomorphisms taking F to F; so let $\bar{H}(\hat{M})$ denote the group of homeomorphisms of $\hat{M}$ that extend to homeomorphisms of M that take F to F modulo those isotopic to the identity (as homeomorphisms of $\hat{M}$ to $\hat{M}$). The main theorem is

THEOREM 1. *The group* $H(M \operatorname{rel} F)$ *is isomorphic to* $\bar{H}(\hat{M})$.

Work of Moise, Bing, Craggs [3] and the author [7] show that homeomorphisms and isotopies of compact 3-manifolds may be approximated via small isotopies by piecewise linear ones. Thus the topological and the piecewise linear theories are identical in the present context. (See [7] for further explanation.) Thus, although I essentially work in the topological category here, I use piecewise linear notions such as regular neighborhood without specifically moving into the piecewise linear category.

3. Lemmas for Theorem 1.

LEMMA 1. *If S is an orientable, closed compact 2-manifold, then* $G(S \times I; S \times \{1\})$ *is path connected.*

Proof. Case 1. *S is a 2-sphere.*

Let N denote the north pole of S and suppose $h \in G(S \times I;\ S \times \{1\})$. It is standard (see, for instance, [13], p. 257) that h is isotopic under an isotopy that leaves $S \times \{1\}$ pointwise fixed to a homeomorphism that leaves $N \times I$ pointwise fixed. Call this new homeomorphism h also. Let D and $E \subset D^\circ$ denote discs in S centered at N such that $E \times I$ is a subset of the interior of $h(D \times I)$. Note that $E \times I$ and $(S \times I) - (D \times I)^\circ$ are 3-balls and to these the Alexander trick [1] may be applied. There is a homeomorphism $\bar{h}$ of $S \times I$ onto itself that is the identity on $(S \times \{1\}) \cup (E \times I)$ and is h outside $D \times I$. The Alexander trick applied first to $(S \times \{0\}) - E^\circ$ and then to the complement of $E^\circ \times I$ yields an isotopy $\{H_\tau\}$ on $S \times I$ such that, for each τ, H_τ is the identity on $(S \times \{1\}) \cup (E \times I)$, $H_1 = \bar{h}$, and $H_0 = \mathrm{id}$.

Now $\bar{h}^{-1} h$ is the identity outside $D \times I$ and on $S \times \{1\}$, so the Alexander trick applied to $D \times I$ yields an isotopy $\{G_\tau\}$ on $S \times I$ such that, for each τ, G_τ is the identity outside $D \times I$ and on $S \times \{1\}$, $G_1 = \bar{h}^{-1} h$ and $G_0 = \mathrm{id}$. Then $\{H_\tau G_\tau\}$ is an isotopy from h to id and $S \times \{1\}$ remains pointwise fixed throughout the isotopy. (This is the Roberts trick [12]; see also [6], p. 213.)

Case 2. *S is not a 2-sphere.*

Waldhausen proved ([17], Lemma 3.5) that every homeomorphism of

$S \times I$ that leaves $S \times \{0\}$ pointwise fixed is isotopic to a level preserving homeomorphism, the isotopy being constant on $S \times \{0, 1\}$. Suppose then that h is level preserving and leaves $S \times \{0\}$ pointwise fixed. (This switch from top to bottom makes the notation simpler.) Then $H_\tau(x, s) = h((x, \tau s), s)$ yields an isotopy from h to the identity that is constant on $S \times \{0\}$.

Corollary. *If S is neither a 2-sphere nor a torus, then if h leaves $S \times \{0, 1\}$ pointwise fixed, then the isotopy can be chosen to be constant on $S \times \{0, 1\}$.*

Proof. The proof rests on the fact that for such S the fundamental group of $G(S)$ is trivial [5]. If h is a level preserving homeomorphism of $S \times I$ that moves no point of $S \times \{0, 1\}$, then there is a loop $\alpha: I \to G(S)$ such that $\alpha(\tau)(x) = y$, where $h(x, \tau) = (y, \tau)$. Since the fundamental group of $G(S)$ is trivial, there is a map $F: I \times I \to G(S)$ such that, for each τ, $F(\tau, 1) = \alpha(\tau)$, $F(\tau, 0) = \mathrm{id}_S$ and, for each s, $F(0, s) = F(1, s) = \mathrm{id}_S$. Then $H_s: S \times I \to S \times I$ defined by $H_s(x, \tau) = \big(F(\tau, s)(x), \tau\big)$ is the required isotopy from h to the identity.

In the following, V is an orientable handlebody with boundary S, and

$$F = \bigcup_{i=1}^{k} ax_i b \quad (k \geqslant 2)$$

is the core of V (as defined in Section 2).

Lemma 2. *The space of homeomorphisms of V that leave S pointwise fixed and take F to itself is path connected.*

Proof. Let $D_1, \ldots, D_k$ be disjoint properly embedded discs in V such that, for each i, $D_i \cap F = x_i$ and let $B_1, \ldots, B_k$ be regular neighborhoods of $D_1, \ldots, D_k$ in V such that, for each i, $B_i \cap F$ is an arc in the interior of $ax_i b$ and $B_i \cap S$ is an annulus A_i. Then $\mathrm{bd}\, B_i - A_i^\circ$ is a union of two properly embedded discs D_i' and D_i'' such that $D_i' \cap F$ is a point x_i' and $D_i'' \cap F$ is a point x_i'' and $\mathrm{cl}(V - \bigcup B_i)$ is a union of two balls K_0 and K_1, $K_j \cap F$ being like the union of k straight-line intervals from the origin of the standard ball to its boundary.

Let h be a homeomorphism of V that leaves S pointwise fixed and takes F to F. Since, for homotopy reasons, h cannot permute the arcs $ax_i b$ if $k \geqslant 3$, there is an isotopy on F from $h|F$ to id_F. If $k = 2$, then F is a circle and the existence of this isotopy is evident. It follows from the isotopy extension theorems of Hudson [9] and the fact that topological homeomorphisms and isotopies may be approximated by piecewise linear ones (see [3] and [7]) that this isotopy extends to an isotopy from h to a homeomorphism that leaves F pointwise fixed, the isotopy being constant on S. Assume then that h leaves F pointwise fixed. Another small isotopy puts $h(D_j)$ in "general position" with respect to $\bigcup (D_i' \cup D_i'')$ so that each component of $h(D_j) \cap \bigcup (D_i' \cup D_i'')$ is a simple closed curve on which $h(D_j)$ crosses $\bigcup (D_i' \cup D_i'')$.

Each such simple closed curve bounds a disc in $h(D_j)$. If D is an innermost such disc, then $D - \mathrm{bd}\, D$ misses $\bigcup(D_i' \cup D_i'')$ and either contains x_j or misses F entirely. If D misses F, then, by standard arguments, D may be moved by an ambient isotopy across a ball and off $\bigcup(D_i' \cup D_i'')$ to reduce the number of components of $h(D_j) \cap \bigcup(D_i' \cup D_i'')$. Suppose then that all innermost discs that miss F have been removed. Then each component of $h(D_j) - \bigcup(D_i' \cup D_i'')$ that misses F is an annulus that links F.

If such an annulus is in some B_i or K_r and has its boundary curves in the same one of D_i' or D_i'', say D_i', then it bounds together with an annulus in D_i' a solid torus [14], so can be moved by an ambient isotopy across the solid torus and off D_i'. If S is not a torus, any such annulus in K_r has, for homotopy reasons, its boundary curves in the same one of D_i' and D_i''. Suppose all these annuli have been removed. Then $h(D_j)$ lies in B_j and standard arguments show that $h(D_j)$ can be moved by an ambient isotopy to D_j. All these isotopies are constant on x_j.

If S is a torus, then at this stage the components of $h(D_j) - \bigcup(D_i' \cup D_i'')$ are "concentric" annuli with boundary curves in different D_i' and D_i'' and a disc E_j containing x_j with boundary in D_j' or D_j'', say D_j'. Then E_j may be pushed off D_j' (if x_j is moved along the simple closed curve F) by an ambient isotopy and this pushing may be repeated until $h(D_j)$ lies in B_j and then is D_j. Do all this for each D_j.

At this stage, h is a homeomorphism that leaves $S \cup F$ pointwise fixed and takes each D_j to D_j and this has been achieved by an isotopy constant on S and on F if S is not a torus. The Alexander trick may now be applied, first to the D_j and then to the components of $V - \bigcup D_j$, to obtain the required isotopy of h to the identity.

COROLLARY. *If S is not a torus, then $G(V; S \cup F)$ is path connected.*

Proof. This follows from the above argument.

LEMMA 3. *If f and g are elements of $G(V \operatorname{rel} F)$ and represent the same element of $H(S)$, then f and g represent the same element of $H(V \operatorname{rel} F)$.*

Proof. There is an isotopy on S from $f|S$ to $g|S$ and, by the isotopy extension theorem (see [9] and [7]), this isotopy extends to an isotopy from f to a homeomorphism f' of V such that $f'|S = g|S$, the isotopy keeping F in F. Lemma 3 now follows from Lemma 2.

4. Proof of Theorem 1. If γ is a homeomorphism of M that takes F to F or of $\hat{M}$ that extends to such a homeomorphism, let $[\gamma]$ denote the isotopy class of γ in $H(M \operatorname{rel} F)$ or in $\bar{H}(\hat{M})$. Suppose that $[\gamma] \in \bar{H}(\hat{M})$. Then γ extends to a homeomorphism $\hat{\gamma}$ in $G(M \operatorname{rel} F)$. Let $\varphi[\gamma] = [\hat{\gamma}]$.

(i) *φ is well defined.*

This follows from Lemma 3.

(ii) *φ is surjective.*

Suppose $[\alpha]\in H(M \operatorname{rel} F)$. For each i, $\alpha(V_i)$ is a regular neighborhood of $\alpha(C_i) = C_{j_i}$ as is V_{j_i}, so by regular neighborhood theory [10] there is an ambient isotopy $\{G_\tau\}$ taking $\alpha(\bigcup V_i)$ to $\bigcup V_i$, $\{G_\tau\}$ being constant on F. Then $\{G_\tau \alpha\}$ is an isotopy taking α to $G_1\alpha$, so $\varphi[G_1\alpha \mid \hat{M}] = [\alpha]$.

(iii) *φ is injective.*

Suppose that $[\gamma]\in \bar{H}(\hat{M})$, that $\hat{\gamma}$ is an extension of γ to an element of $G(M \operatorname{rel} F)$, and that $\hat{\gamma}$ is isotopic to the identity under an isotopy $G = \{G_\tau\}$ that leaves F in F. Then $\gamma(V_i) = V_i$ for each i. Let V_i' be a regular neighborhood of C_i that misses all $G_\tau(S_i)$ and let V_i'' be a regular neighborhood of V_i. Denote the boundaries of V_i' and V_i'' by S_i' and S_i'', respectively. The isotopy $G \mid (M - \bigcup V_i) \times I$ extends to an isotopy

$$G'\colon \operatorname{cl}(M - \bigcup V_i') \times I \to M$$

such that $G'(S_i') = S_i'$ for each τ and i and

$$G_0' = G_0 \mid \operatorname{cl}(M - \bigcup V_i') \times I.$$

Then G' extends to an isotopy $G''\colon M \times I \to M$ such that $G_0'' = G_0$ and, for each τ, $G_\tau''(F) = F$. (This follows from the isotopy extension theorem and Lemma 3.)

By Lemma 1, $G(\operatorname{cl}(V_i - V_i''); S_i)$ is path connected. Therefore, it may be assumed first that G_1' is the identity and then (Lemma 2) that G_1'' is the identity. Now for each τ the product structure on $G''(\operatorname{cl}(V_i - V_i'))$ and $G''(\operatorname{cl}(V_i'' - V_i))$ induced via G'' by that on $\operatorname{cl}(V_i - V_i'')$ and $\operatorname{cl}(V_i'' - V_i)$ may be used to an ambient isotopy $G_\tau(S_i)$ down to S_i' and then back up to S_i, the isotopy constant outside $\bigcup V_i''$. This yields an isotopy $H\colon M \times I \to M$ such that, for each τ,

$$H_\tau \mid (M - V_i'') = G_\tau \mid (M - V_i''), \qquad H_\tau(V_i) = V_i,$$

$$H_\tau(C_i) = C_i, \qquad H_0 = G_0, \qquad \text{and} \qquad H_1 = \mathrm{id}_M$$

($G_1'' = \mathrm{id}$, so $H_1 = \mathrm{id}$ by construction). The $H \mid \hat{M} \times I$ is an isotopy of γ to the identity.

(iv) *φ is a homomorphism.*

Clearly, if $[\gamma], [\gamma'] \in \bar{H}(\hat{M})$, and $\hat{\gamma}$ and $\hat{\gamma}'$ represent $\varphi[\gamma]$ and $\varphi[\gamma']$, respectively, then $\hat{\gamma}\circ\hat{\gamma}' = \gamma\circ\gamma'$ on each S_i, so $\hat{\gamma}\circ\hat{\gamma}'$ extends $\gamma\circ\gamma'$ to a homeomorphism of M that takes F to F. Thus $\hat{\gamma}\circ\hat{\gamma}' \in \varphi[\gamma\circ\gamma']$, so $\varphi[\gamma\circ\gamma'] = \varphi[\gamma]\circ\varphi[\gamma']$. This completes the proof of Theorem 1.

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