COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

CONCERNING EXPANSIVE POINT SETS AND PROPERTIES RELATED TO NORMALITY

BY

A. R. VANCLEAVE, Jr. (AUBURN, ALABAMA)

In this paper five notions of expansive point set are defined. These are used to give characterizations of those topological T_1-spaces which are normal, hereditarily normal, collectionwise normal, and hereditarily collectionwise normal.

The word "space" will mean topological T_1-space. The closure of M in the space S will be denoted by $\mathrm{Cl}(M)$, and the closure of M in a subspace K of S will be denoted by $\mathrm{Cl}(M, K)$. The boundary of M will be denoted by $\mathrm{B}(M)$ and the interior by M^0. Domain is synonymous with open set, and if G is a collection of sets, G^* will denote the sum of the elements of G.

Definition 1. The point set M in the space S is said to be *expansive* (*c-expansive*) in S provided that if G' is a (countable) collection of mutually exclusive domains with respect to M, then there is a collection G of mutually exclusive domains in S such that each element of G contains only one element of G' and G covers G'^*.

Definition 2. The point set M in the space S is said to be *w-expansive* (*wc-expansive*) in S provided that if G' is a (countable) discrete collection of domains with respect to M, then there is a collection G of mutually exclusive domains in S such that each element of G contains only one element of G' and G covers G'^*.

Definition 3. The point set M in the space S is said to be *q-expansive* in S provided that if h and k are mutually exclusive domains with respect to M, then there exist mutually exclusive domains D and E in S such that $h \subset D$ and $k \subset E$.

THEOREM 1. *The space S is normal if and only if each closed subset of S is wc-expansive in S.*

Proof. Suppose S is normal and M is closed in S. Let $G' = \{g_1, g_2, \ldots\}$ denote a discrete countable collection of domains with respect to M. Then $\{\mathrm{Cl}(g_1), \mathrm{Cl}(g_2), \ldots\}$ is a discrete countable collection of closed sets in S. Hence there is a collection of mutually exclusive domains, D_1, D_2,

$D_3, \ldots$, in S such that for each n, $D_n \supset \mathrm{Cl}(g_n) \supset g_n$. Hence M is wc-expansive in S. Now suppose that each closed subset of S is wc-expansive in S. Let H and K denote disjoint closed sets in S. Then $H+K$ is closed and $\{H, K\}$ is a discrete collection of domains with respect to $H+K$. Hence there are mutually exclusive domains D and E in S containing H and K, respectively.

LEMMA 1. *Suppose $J \subset S$ and G' is a collection of mutually exclusive domains with respect to J. For each g' in G', let g denote a domain in S such that $g \cdot J = g'$. Let $G = \{g \mid g' \epsilon G'\}$. Then in the space G^*, $M = \{\mathrm{Cl}(g', G^*) \mid g' \epsilon G'\}$ is a discrete collection of closed point sets.*

Proof. Suppose J, G', G, and M are as above. Let $g'' = \mathrm{Cl}(g', G^*)$ so that $M = \{g'' \mid g' \epsilon G'\}$.

Now M is a collection of mutually exclusive sets. For suppose that g'' and h'' are two elements of M and that p is a point in $g'' \cdot h''$. Since $p \epsilon G^*$, p is in some element of G. Suppose $p \epsilon g$. If $g \cdot h''$ exists, then $g \cdot h'$ exists and $g \cdot J \neq g'$. Hence g cannot intersect h'' and so p cannot be in both g and $g'' \cdot h''$. Hence p is in some element, k, of G. Since k is a domain in G^* containing p, k must contain a point, q, of g'. But $q \epsilon g'$ implies that $q \epsilon J$ and hence $q \epsilon J \cdot k = k'$. Then $q \epsilon g' \cdot k'$, which contradicts the fact that G' is a collection of mutually exclusive sets. Hence no two elements of M intersect.

Also, M is a discrete collection in G^*. For suppose that L is a subcollection of M and that p is a limit point of L^* in G^*. Since $p \epsilon G^*$, p is contained in some element, k, of G. Since k is open in G^*, k intersects L^*. If k intersects some element h'' of L, then k intersects h' and $h' = k'$ since G' is a disjoint family of sets. Hence k intersects only one element of L, namely k''. Therefore $p \epsilon k'' \subset L^*$ and $\mathrm{Cl}(L^*, G^*) = L^*$. Hence M is discrete.

THEOREM 2. *The following statements are equivalent:*

(1) *The space S is hereditarily normal.*

(2) *Each open subset of S is normal.*

(3) *Each subset of S is c-expansive in S.*

(4) *Each subset of S is wc-expansive in S.*

(5) *Each subset of S is q-expansive in S.*

(6) *Each closed subset of S is q-expansive in S.*

Proof. The implications (1) $\to$ (2), (3) $\to$ (4), (3) $\to$ (5), and (5) $\to$ (6) follow directly from the definitions.

(2) $\to$ (3). Suppose each open subset of S is normal. Also suppose that J is a subset of S and that G' is a countable collection of mutually exclusive domains with respect to J. For each g' in G' let g be a domain in S such that $g \cdot J = g'$. Let $G = \{g \mid g' \epsilon G'\}$. Let $M = \{\mathrm{Cl}(g', G^*) \mid g' \epsilon G'\}$. By Lemma 1, M is a discrete countable collection of closed point sets

in the space G^*. Also G^* is open in S, and by hypothesis G^* is normal. Hence there is a collection, H, of mutually exclusive domains with respect to G^* such that each element of H contains only one element of M and H covers M^*. Also, each element of H is open in S so that J is c-expansive in S.

(4) $\rightarrow$ (1). Suppose each subset of S is wc-expansive in S. Let J be a subspace of S and let H and K denote disjoint closed sets with respect to J. Let $L = H + K$. Then $\{H, K\}$ is a discrete collection of domains with respect to L. Since L is wc-expansive in S, there are disjoint domains D and E in S containing H and K, respectively. Then $D' = D \cdot J$ and $E' = E \cdot J$ are disjoint domains with respect to J containing H and K, respectively. Hence J is normal.

Thus the first four statements are equivalent.

(6) $\rightarrow$ (2). Suppose each closed subset of S is q-expansive in S. Also suppose that U is open in S and that H and K are disjoint closed sets with respect to U. Let $L = \mathrm{Cl}(H + K)$. Now H is a domain with respect to L since $H = (U - K) \cdot L$ and $U - K$ is open in S. Similarly K is a domain with respect to L. Since L is closed, L is q-expansive in S and there are disjoint domains D and E in S containing H and K, respectively. Then $D' = D \cdot U$ and $E' = E \cdot U$ are disjoint domains with respect to U containing H and K, respectively. Hence U is normal.

That statements (1) and (2) are equivalent was previously shown by Dowker [2].

THEOREM 3. *The following statements are equivalent:*

(1) *The space S is collectionwise normal.*

(2) *The boundary of each domain in S is w-expansive in S.*

(3) *Each closed subset of S is w-expansive in S.*

Proof. That (3) $\rightarrow$ (2) is obvious.

(2) $\rightarrow$ (1). Suppose that the boundary of each domain in S is w-expansive in S and that M is a discrete collection of closed point sets. Let $M_1 = \{m \mid m \in M, m = m^0\}$ and let $M_2 = M - M_1$. For each $m \in M_2$, let $m' = m - m^0$ and let $V = [S - M_2^*] + \sum m^0$ $(m \in M_2)$. Then V is a domain and $B(V) = \sum m'$ $(m \in M_2)$ is w-expansive in S. Also $\{m' \mid m \in M_2\}$ is a discrete collection of domains with respect to $B(V)$. Hence there is a collection G of mutually exclusive domains in S such that each element of G contains only one element of $\{m' \mid m \in M_2\}$ and G covers $\{m' \mid m \in M_2\}^*$. Denote the element of G containing m' by $g_{m'}$. For each m in M_2 let $z_m = \sum h$ $(h \in M, h \neq m)$ and let $g_m = [g_{m'} + m^0] \cdot [S - z_m]$. For each m in M_1 let $g_m = m$. Then for each m in M, g_m is open and contains m. Hence $H = \{g_m \mid m \in M\}$ is a collection of mutually exclusive domains in S such that each element of H contains only one element of M and H covers M^*. Hence S is collectionwise normal.

(1) $\to$ (3). Suppose that S is collectionwise normal, M is closed in S, and G' is a discrete collection of domains with respect to M. Then $G'' = \{\mathrm{Cl}(g') \mid g' \epsilon G'\}$ is a discrete collection of closed point sets in S. Then there is a collection, G, of mutually exclusive domains in S such that each element of G contains only one element of G'' and G covers G''^*. Therefore M is w-expansive in S.

THEOREM 4. *The following statements are equivalent*:

(1) *The space S is hereditarily collectionwise normal.*

(2) *Each open subset of S is collectionwise normal.*

(3) *Each subset of S is expansive in S.*

(4) *Each subset of S is w-expansive in S.*

(5) *Each closed subset of S is expansive in S.*

Proof. That (1) $\to$ (2) and (3) $\to$ (4) follows from the definitions.

(2) $\to$ (3). Suppose that each open subset of S is collectionwise normal. Suppose further that $J \subset S$, and G' is a collection of mutually exclusive domains with respect to J. For each g' in G' let g be a domain in S such that $g \cdot J = g'$. Let $G = \{g \mid g' \epsilon G'\}$. By Lemma 1, $M = \{\mathrm{Cl}(g', G^*) \mid g' \epsilon G'\}$ is a discrete collection of closed point sets in the space G^*. By hypothesis G^* is collectionwise normal, so there is a collection, H, of mutually exclusive domains with respect to G^* such that each element of H contains only one element of M and such that H covers M^*. But the elements of H are open in S so J is expansive in S.

(4) $\to$ (1). Suppose each subset of S is w-expansive in S. Also suppose that $J \subset S$ and M is a discrete collection of closed point sets with respect to J. Consider the subspace M^* of S. If $m \epsilon M$, then m is a domain with respect to M^* since $m = M^* - \sum h$ $(h \epsilon M, h \neq m)$. Hence $\{m \mid m \epsilon M\}$ is a discrete collection of domains with respect to M^*. Since M^* is w-expansive in S there is a collection, G, of mutually exclusive domains in S such that each element of G contains only one element of M and G covers M^*. Let $K = \{g \cdot J \mid g \epsilon G\}$. Then K is the desired collection of domains with respect to J and J is collectionwise normal.

We have shown that the first four statements are equivalent. That (3) $\to$ (5) is obvious.

(5) $\to$ (2). Suppose that each closed subset of S is expansive in S. Let U denote an open set in S. Let $M = \{m_a \mid a \epsilon A\}$ denote a discrete collection of closed point sets with respect to U which is faithfully indexed by A. Let $L = \mathrm{Cl}[\sum m_a (a \epsilon A)]$ and for each a in A let $z_a = \sum m_b (b \epsilon A$, $b \neq a)$. Hence if $a \epsilon A$, $U - z_a$ is open in S. Also $m_a = (U - z_a) \cdot L$, so m_a is a domain with respect to L. Then $\{m_a \mid a \epsilon A\}$ is a collection of mutually exclusive domains with respect to L. Since L is expansive in S there is a collection $\{v_a \mid a \epsilon A\}$ of mutually exclusive domains in S such that for each $a \epsilon A$, $m_a \subset v_a$.

For each $a \in A$, let $g_a = v_a \cdot U$. Then $\{g_a | a \in A\}$ is a collection of mutually exclusive domains with respect to U, and for each $a \in A$, $m_a \subset g_a$. Therefore U is collectionwise normal.

That (2) → (1) was previously shown by Šedivá [3]. Aull [1] has announced a theorem that shows (1) → (4).

REFERENCES

[1] C. E. Aull, *Collectionwise normal subsets*, Notices of the American Mathematical Society 14 (1967), p. 843.

[2] C. H. Dowker, *On inductive dimension of completely normal spaces*, Quarterly Journal of Mathematics 4 (1953), p. 267-281.

[3] V. Šedivá, *On collectionwise normal and hypocompact spaces*, Czechoslovak Mathematical Journal 84 (1959), p. 50-62.

AUBURN UNIVERSITY

Reçu par la Rédaction le 1. 2. 1968