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ON COUNTABLE LOCALLY CONNECTED SPACES

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It is known that examples of countable locally connected Hausdorff spaces are rare and most of them are obtained by artificial constructions. In a recent paper, Jones and Stone [3] constructed countable connected, locally connected T_a-spaces for each countable ordinal a. (Here the separation axiom T_a is stronger than the usual Hausdorff axiom for $a > 1$.) They asked (see [3], P 707) whether the Urysohn space constructed by them is homogeneous. In section 2, we give a negative answer.

In section 1, we give one more collection of examples of such spaces. We prove that the set of all strictly increasing sequences of a finite odd length in a connected space becomes a locally connected space under some mild conditions and a curious topology.

We deduce that, for each ordinal a, there are plenty of countable connected, locally connected T_a-spaces — as many as there are countable spaces.

For a detailed information about local connectedness in countable spaces, we refer to [4].

1. A locally connected topology on a set of finite sequences. Let X be a countable connected Hausdorff space such that $X\setminus A$ is connected whenever A is a finite set (e.g., X may be the space of Bing [1] or the space of Brown [2]; in fact, there are plenty of such spaces (cf. [4])). By a pre-assigned one-to-one correspondence of X with the set of natural numbers, we give a well ordering for X. This allows us to talk of increasing sequences of elements in X. We denote by $L(X)$ the set of all those strictly increasing finite sequences of elements in X for which the length is odd. We shall presently prove that $L(X)$ is a connected locally connected extension of X under a well-specified topology. We shall also show that many nice topological properties are preserved by this extension process.

Note that a general element of $L(X)$ is of the form

$$a = (x_1, x_2, \ldots, x_{2n+1}),$$

where each $x_i \in X$ and $x_1 < x_2 < \ldots < x_{2n+1}$. Let U be a basic neighbourhood of x_{2n+1} in X. Then we define a subset αU of $L(X)$ as follows: αU is the set of all elements of the form

$$(x_1, x_2, \ldots, x_{2n}, y_1, y_2, \ldots, y_{2m+1})$$

satisfying one of the following two conditions:

(1) $y_{2m+1} \in U$ and, for each $1 \leqslant l \leqslant m$, at least one of the two terms y_{2l-1} and y_{2l} belongs to U;

(2) there exists an integer k with $1 \leqslant k \leqslant m$ such that both y_{2k-1} and y_{2k} belong to U and such that, for each l with $1 \leqslant l \leqslant k$, at least one of the two terms y_{2l-1} and y_{2l} belongs to U.

The following are easily noted:

1. $\alpha \in \alpha U$.
2. If $V \subset U$, then $\alpha V \subset \alpha U$.
3. If $\beta \in \alpha U$ and is of the first type, then $\beta U \subset \alpha U$.
4. If $\beta \in \alpha$ and is of the second type, then $\beta V \subset \alpha U$ for any V for which βV is defined.

These facts show that $\{\alpha U \mid \alpha \in L(X);\ U$ is a basic neighbourhood of the last term of $\alpha\}$ is a base for a topology on $L(X)$. This is the topology that will be proved to have the above-mentioned properties.

PROPOSITION 1. *$L(X)$ is a Hausdorff space.*

Proof. Let $\alpha = (x_1, x_2, \ldots, x_{2n+1})$ and $\beta = (y_1, y_2, \ldots, y_{2m+1})$ be any two distinct elements of $L(X)$. We want to show that they can be separated by disjoint open sets. We assume, without loss of generality, that $n \leqslant m$.

Case 1. Let $x_i \neq y_i$ for some $i \leqslant 2n$. Then αU and βV are disjoint for any choice of U and V for which they are defined.

Case 2. Let $m > n$ and let neither y_{2n+1} nor y_{2n+2} be equal to x_{2n+1}. Then choose a basic neighbourhood U of x_{2n+1} that avoids both y_{2n+1} and y_{2n+2}. Then αU and βV must be disjoint for any choice of V such that βV is defined.

Case 3. Let $x_{2n+1} \neq y_{2m+1}$, i.e., the last terms of α and β are different. Then choose two disjoint open sets U and V that are neighbourhoods of x_{2n+1} and y_{2m+1}, respectively. We claim that αU and βV are disjoint. Let $\gamma \in \alpha U \cap \beta V$ if possible, say, $\gamma = (t_1, t_2, \ldots, t_{2s+1})$. Since U and V are disjoint, at least one of them must avoid t_{2s+1}, say, $t_{2s+1} \notin U$. Since $\gamma \in \alpha U$, this implies the existence of an integer k such that $1 \leqslant k \leqslant s$ and t_{2k-1} and t_{2k} are both in U and such that at least one of t_{2l-1} and t_{2l} is in U for each $1 \leqslant l \leqslant k$. This, in turn, means that, for each l, $1 \leqslant l \leqslant k$, it is not true that both t_{2l-1} and t_{2l} are in V; moreover, it also implies that neither t_{2k-1} nor t_{2k} is in V. These results imply that γ cannot be in βV, a contradiction.

Now we show that these three cases exhaust all possibilities. If $m = n$, it is clear that either case 1 or case 2 must hold. So, let $m > n$. If case 2 does not hold, then either $y_{2n+1} = x_{2n+1}$ or $y_{2n+2} = x_{2n+1}$. In either case, y_{2m+1} must be greater than x_{2n+1} and so case 3 occurs.

PROPOSITION 2. *Let* $\omega = (x_1, x_2, \ldots, x_{2n})$ *be a strictly increasing sequence of elements in* X *that has the even length. Then there exists a map* f_ω *from the set*

$$A_\omega = \{x_{2n-1}\} \cup \{x \in X \mid x \geqslant x_{2n}\}$$

to $L(X)$ *such that* f_ω *is a homeomorphism of* A_ω *onto a subspace of* $L(X)$.

Proof. Take

$$\begin{aligned} f_\omega(x_{2n-1}) &= (x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}),\\ f_\omega(x_{2n}) &= (x_1, x_2, \ldots, x_{2n-2}, x_{2n}),\\ f_\omega(x) &= (x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}, x_{2n}, x) \quad \text{if } x \notin \{x_{2n-1}, x_{2n}\}. \end{aligned}$$

Clearly, f_ω is one-to-one.

The continuity of f_ω follows from the observation that, for each basic open neighbourhood aU of a in $f_\omega(A_\omega)$, it is true that $U = f_\omega^{-1}(aU)$. The openness of f_ω follows from the observation that if U is a neighbourhood of x_{2n-1} or x_{2n}, then

$$f_\omega(U) = aU \cap f_\omega(A_\omega),$$

where a equals $f_\omega(x_{2n-1})$ or $f_\omega(x_{2n})$, respectively; and if U is a neighbourhood of x ($\notin \{x_{2n-1}, x_{2n}\}$) not containing x_{2n-1} or x_{2n}, then

$$f_\omega(U) = aU \cap f_\omega(A_\omega), \quad \text{where } a = f_\omega(x).$$

Since the set A_ω is connected (by the assumption, the complement of any finite set in X is connected), we have

COROLLARY 3. *The range of* f_ω *is connected.*

PROPOSITION 4. *The space* $L(X)$ *is connected.*

Proof. First, note that if $a = (x_1, x_2, \ldots, x_{2n+1})$ is an arbitrary element of $L(X)$, then it follows from the previous proposition that there is a connected set containing a and the point $(x_1, x_2, \ldots, x_{2n-1})$. Repeating the argument n times, we see that the connected component of a contains the point (x_1). But we easily see that the map $x \mapsto (x)$ is a homeomorphism of X onto a subspace of $L(X)$, and so its range is connected. All these facts together prove that if $(x_0) \in L(X)$ is fixed, then every point of $L(X)$ lies in the connected component of (x_0). Hence $L(X)$ is connected.

PROPOSITION 5. *Every basic open set aU of $L(X)$ is connected.*

Proof. First, we note that if

$$\gamma_1 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m}, u_1)$$

and

$$\gamma_2 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m}, u_2)$$

are two elements of aU in this form, where u_1 and u_2 belong to U and $u_1 < u_2$, then they lie in the same connected component of aU. This follows from Proposition 2 when we observe that

$$f_\omega(A_\omega) \subset aU \quad \text{if } \omega = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m}, u_1, u_2).$$

Secondly, if

$$\gamma_1 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m}, u_1)$$

and

$$\gamma_3 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m-1}),$$

and if $y_{2m-1} \in U$, then γ_1 and γ_3 are in the same component of aU. This follows from the fact that γ_3 is in the closure of the set of all elements of the above-given form γ_2. The similar assertion holds for γ_4 and γ_1, where

$$\gamma_4 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m-2}, y_{2m}).$$

Thirdly, if

$$\delta = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m+1}) \in aU$$

is such that $y_{2m+1} \in U$, then there exists an integer k such that both y_{2k-1} and y_{2k} belong to U. If we set

$$\omega = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m}),$$

it follows that $f_\omega(A_\omega) \subset aU$ and so δ and

$$\delta_1 = (x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2m-1})$$

are in the same connected component of aU.

These three facts together with the principle of induction prove that any element of aU belongs to the connected component of a in aU. Thus aU is connected.

Let α be an ordinal number. A topological space X is said to be a T_α-*space* if, whenever x and y are distinct elements of X, there exists a transfinite sequence $\{U_\beta \mid \beta \leqslant \alpha\}$ of open sets such that $x \in U_1$, $y \in X \setminus U_\alpha$ and $\bar{U}_\beta \subset U_\gamma$ whenever $\beta < \gamma$.

THEOREM 6. *Let X be any countable connected Hausdorff space in which every finite-complement subset is connected. Then the space $L(X)$ of all strictly increasing finite sequences of odd length of elements in X is a connected, locally connected Hausdorff space. Further,*

(1) *X and $L(X)$ have the same cardinality, weight and local weight;*

(2) *if X satisfies the separation axiom T_a for some ordinal a, then so does $L(X)$, and conversely;*

(3) *X is homeomorphic to a closed subspace of $L(X)$;*

(4) *X is regular at a point if and only if $L(X)$ is regular there.*

Proof. We can easily prove (1). The map $x \mapsto (x)$ can be seen to be a homeomorphic embedding of X in $L(X)$ and its range is easily verified to be closed in $L(X)$. (2) and (4) can be proved by straightforward methods.

Remark. Jones and Stone [3] have proved that, for each countable ordinal a, there exists a countable connected, locally connected T_a-space. We can prove

THEOREM 7. *Let a be any countable ordinal. Then there exist 2^c distinct topological types of countable connected, locally connected T_a-spaces.*

Proof. Let X be any countable regular space and let a be any countable ordinal. Since X is zero-dimensional, it is a T_a-space. For each pair of points x, y in X, take a copy $X_{x,y}$ of X with $f_{x,y}\colon X \to X_{x,y}$ a homeomorphism. Keep X and the copies $X_{x,y}$ pairwise disjoint and then identify pairs of points: for each x in X, x is identified with $f_{x,y}(x)$ for every y in X; similarly, each y in X is identified with $f_{x,y}(y)$ for each x in X. Denote the new space by X_1. Note that X is embedded in X_1.

Repeat the same process with X_1 in place of X. We get a bigger space X_2.

Thus, by induction, we get a direct limit of spaces X_n and homeomorphic embeddings of X_m in X_n whenever $m \leqslant n$. Let Y be this direct limit. Then it can be proved that Y is a countable connected T_a-space containing X as a subspace and that if A is any finite subset of Y, then $Y \setminus A$ is connected.

Let $L(Y)$ be the space constructed from Y as described earlier. Then $L(Y)$ is a countable connected, locally connected T_a-space containing X as a subspace.

Thus we have shown that every countable regular space is a subspace of a countable connected, locally connected T_a-space, where a is an arbitrary countable ordinal.

The assertion of the theorem now follows from the fact that there exist 2^c mutually non-homeomorphic countable regular spaces, whereas a countable space can have at most c types of subspaces.

2. On a question of Jones and Stone. Jones and Stone [3] constructed an example of a connected locally connected Urysohn space X. They asked (cf. [3], Problem 707) whether their space X is homogeneous. They guessed that the odd points of X look different from the even points of X and hence expected a negative answer. In this section we prove that the answer is negative, as they expected, but not because of the difference between odd and even points of X. We show that the points of lowest level are different from the points of higher levels.

For the sake of completeness, we include a description of their space X here. First, we describe the space $S(a, b)$. Let $\{X_n \mid n = 0, \pm 1, \pm 2, \ldots\}$ be a collection of disjoint subsets of the real line R such that each X_n is a countable dense subspace of R. Let the underlying set of $S(a, b)$ be

$$\{a, b\} \cup \Big(\bigcup_{n=-\infty}^{\infty} X_n \Big),$$

where a and b are two extra points. For each $\varepsilon > 0$, write

$$D_\varepsilon(x) = \begin{cases} X_n \cap (x-\varepsilon, x+\varepsilon) & \text{if } x \in X_n \text{ and } n \text{ is even,} \\ (X_{n-1} \cup X_n \cup X_{n+1}) \cap (x-\varepsilon, x+\varepsilon) & \text{if } x \in X_n \text{ and } n \text{ is odd,} \\ \bigcup_{n>1/\varepsilon} X_n & \text{if } x = a, \\ \bigcup_{n<1/\varepsilon} X_n & \text{if } x = b. \end{cases}$$

Then these $D_\varepsilon(x)$'s define the neighbourhood bases of a connected Hausdorff topology on the countable set $S(a, b)$.

We take $G_0 = E_0$, homeomorphic to $S(a, b) \setminus \{a, b\}$.

For each pair (p, q) of points of E_0, we take a copy $E_1(p, q)$ of the space $S(a, b)$ and identify its special points with p and q, respectively.

We let G_1 be the union of E with all these $E_1(p, q)$'s attached as above.

Suppose we have already defined G_n. To each pair (p, q) of points in $G_n \setminus G_{n-1}$ such that $p, q \in E_n(r, s)$ for some $r, s \in G_{n-1}$, we attach a copy $E_{n+1}(p, q)$ of $S(a, b)$, identifying its special points with p and q. We let G_{n+1} to be the union of G_n with all these $E_{n+1}(p, q)$'s attached as above.

We let X to be the union of all G_n's. For each $x \in X$, there is a unique n such that $x \in G_n \setminus G_{n-1}$ and a unique copy $E_n(p, q)$ of $S(a, b)$ such that p and q belong to G_{n-1} and $x \in E_n(p, q)$.

Given $\varepsilon > 0$, we define $N_\varepsilon(x)$ to be the smallest subset of X such that

(i) $N_\varepsilon(x) \supset D_\varepsilon(x)$ in $E_n(p, q)$;

(ii) if $i \geqslant n$ and r and $s \in N_\varepsilon(x)$, then $E_i(r, s) \subset N_\varepsilon(x)$;

(iii) if $i \geqslant n$ and $r \in N_\varepsilon(x)$, then $N_\varepsilon(x) \supset D_\varepsilon(r)$ in every copy of the form $E_i(r, s)$ homeomorphic to $S(a, b)$.

These $N_\varepsilon(x)$'s define a connected locally connected Urysohn topology on X.

THEOREM 8. *The countable connected, locally connected Urysohn space X constructed by Jones and Stone* [3] *is not homogeneous.*

Proof. We show that the elements of E_0 are unlike the elements of $X \setminus E_0$. More precisely, we show that if $x_1 \in E_0$ and $x_2 \in X \setminus E_0$, then no self-homeomorphism of X can take x_1 to x_2.

It is obvious from the construction of X that each element $x \in X \setminus E_0$ belongs to some $E_i(p, q)$, where $E_i(p, q)$ is homeomorphic to $S(a, b)$ and is not locally connected at x. We show that no $x \in E_0$ has this property. In other words, we show that

If $x \in E_0$ and if $A \subset X$ is such that $x \in A$ and A is homeomorphic to $S(a, b)$, then A is locally connected at x.

In order to prove this, we introduce a simple notation for the sake of convenience in the proof. If y_1 and y_2 are two distinct elements in E_0, then we define $\{y_1, y_2\}^*$ to be the set of all elements that lie strictly above them in X. More precisely,

$$\{y_1, y_2\}^* = \bigcup \{N_\varepsilon(y) \mid \varepsilon > 0;\, y \in E_1(y_1, y_2)\}.$$

Analogously, we define $\{z_1, z_2\}^*$ for each pair of distinct elements z_1 and z_2 that lie in $E_1(y_1, y_2)$.

Now to the proof of our claim. Since A is homeomorphic to $S(a, b)$, it has exactly two points, say, p and q, where it is locally connected. We want to show that x is one of them. Supposing the contrary, we show that we are led to contradictions.

First, we observe that $A \setminus E_0$ is non-empty, since A is connected whereas E_0 is totally disconnected. Secondly, we note that $A \setminus E_0$ is open in A, since E_0 is closed in X. These together imply that $A \setminus E_0$ is infinite. Now

$$X \setminus E_0 = \bigcup_{y_1, y_2 \in E_0} \{y_1, y_2\}^*.$$

Therefore, $A \setminus E_0$ must meet $\{y_1, y_2\}^*$ for some $y_1 \neq y_2$ in E_0.

Suppose $A \setminus E_0$ meets $\{y_1, y_2\}^*$, where $y_1, y_2 \in E_0$, $y_1 \neq y_2$. Then consider $A \setminus \{y_1, y_2\}$. If $\{y_1, y_2\} \neq \{p, q\}$, this is connected and hence contained in $\{y_1, y_2\}^*$, since $\{y_1, y_2\}^*$ is closed and open in $X \setminus \{y_1, y_2\}$.

This last observation will be used more than once in what follows. We divide the proof into two cases, in both of which we arrive at contradictions.

Case 1. Let both p and q belong to E_0. Then, by the above-mentioned observation, $A \setminus E_0$ cannot meet $\{y_1, y_2\}^*$ for any $\{y_1, y_2\} \neq \{p, q\}$, where $y_1, y_2 \in E_0$. Therefore, $A \subset E_0 \cup \{p, q\}^*$. By our assumption, $x \notin \{p, q\}$. Now, E_0 is zero-dimensional. Therefore, there exists a closed and open neighbourhood W of x in E_0 which avoids both p and q. Then $A \cap W$ is closed in A, since W is closed in E_0 and hence in X. Also $E_0 \cup \{p, q\}^* \setminus W$ is closed in X and so $A \setminus W$ is closed in A. Thus $A \cap W$ is closed and open in A. It is neither empty (since $x \in A \cap W$) nor the whole A (since $p \notin A \cap W$). This contradicts the connectedness of A.

Case 2. Let case 1 do not hold (i.e., at least one of the elements in $\{p, q\}$ is in $A \setminus E_0$). Then, by the observation, we see that there exist y_1, y_2 in E_0 such that $A \setminus \{y_1, y_2\} \subset \{y_1, y_2\}^*$. Since $x \in A$, this implies that $x \in \{y_1, y_2\}$. Thus $A \subset \{x, y\} \cup (x, y)^*$ for some $y \in E_0$.

Now we repeat our argument in the second level. First, observe that $E_1(x, y) \setminus \{x, y\}$ is totally disconnected, but $A \setminus \{x, y\}$ is connected. Therefore, $A \setminus E_1(x, y)$ is non-empty. It is open in A, since $E_1(x, y)$ is closed in X. Therefore, it is infinite, since A is connected.

Analogously to the observation, we see that if $z_1, z_2 \in E_1(x, y) \setminus \{x, y\}$ and if $A \setminus E_1(x, y)$ meets $\{z_1, z_2\}^*$, then either $\{z_1, z_2\} = \{p, q\}$ or $A \subset \{z_1, z_2\} \cup \{z_1, z_2\}^*$. But the second case is impossible, since $x \in A$. Therefore, we get that both p and q belong to $E_1(x, y)$ and $A \setminus E_1(x, y)$ is contained in $\{p, q\}^*$. Now, the set $A \setminus \{x, y\}$ is connected and is contained in $C \cup \cup \{p, q\}^*$, where C is the zero-dimensional set $E_1(x, y) \setminus \{x, y\}$. Arguing as at the end of case 1, we see that if C is non-empty, we are led to a contradiction with the connectedness of $A \setminus \{x, y\}$. Therefore, C must be empty and hence $A \setminus \{x, y\} \subset \{p, q\}^*$. This is again impossible, since x would then be an isolated point of A.

Thus, in both cases, the assumption that $x \notin \{p, q\}$ leads to contradiction. Therefore, $x \in \{p, q\}$ and this completes the proof of our claim.

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