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ACTIONS OF Z_n ON SOME SURFACE-BUNDLES OVER S^1

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1. Introduction. In this paper we consider the classification problem of Z_n-actions on manifold-bundle over the circle. We solve this problem for actions which are free except, possibly, at a finite number of points. We assume also that the fibers are equal to S^1, S^2, P^2, D^1 or D^2. We extend results of Tao [16], Ritter [14], and Tollefson [17] concerning the action of Z_n on $S^1\times S^2$, results of Kim [4] on the involution of N, and results of Kim [5] on actions of Z_n on $S^1\times D^2$.

We work in the PL-category. All manifolds are assumed to be connected, unless otherwise specified. We adopt the terminology of [3]. Let N denote a nonorientable S^2-bundle over S^1, B a Klein bottle, Bs a solid Klein bottle, and Mb a Möbius band.

1.1. Definition. Let $M_\varphi = S^1\hat{\times} F = R\times F/\sim$, $(t, y)\sim (t+1, \varphi(y))$, be an F-bundle over S^1, where F is a manifold and φ is a self-homeomorphism of F. An action of Z_n (generated by T) on M_φ is said to be *standard* if it may be described by one of the following expressions:

1. $T(t, y) = (t+i/s, g_0(y))$, where g_0 is a self-homeomorphism of F, s divides n, $0<i\leqslant s$, $(i,s)=1$ (i.e., i is relatively prime to s), $g_0^n = \varphi^{ni/s}$, and $g_0\varphi = \varphi g_0$. We call such an action a *standard action of type* $(1;\, n, s, i, g_0)$ or, more exactly, $(1;\, n, s, i, g_0)_{(F,\varphi)}$.

2. $T(t, y) = (1-t, g_0(y))$, where n is even, g_0 generates a Z_n-action on F, and $\varphi g_0 = g_0\varphi^{-1}$. We call such an action a *standard action of type* $(2;\, n, g_0)$ or, more exactly, $(2;\, n, g_0)_{(F,\varphi)}$.

1.2. Remark. An action of type $(1;\, n, s, i, g_0)$ on M_φ is free iff $g = \varphi^{-i}g_0^s$ generates a free action on $Z_{n/s}$ on F (in particular, if $g = \mathrm{Id}$ and $s = n$). An action of type $(2;\, n, g_0)_{(F,\varphi)}$ is free iff g_0 and $g_0\varphi$ generate free actions of Z_n on F.

1.3. Definition. Two G-actions μ_1, μ_2: $G\times M\to M$ are called *weakly conjugate* if there exist a group automorphism A: $G\to G$ and a self-homeomorphism $f: M \to M$ (f preserves orientation if M is orientable) such that $\mu_1 = f^{-1}\mu_2(A \times f)$. If A is the identity, then μ_1 and μ_2 are conjugate.

Let $\mathrm{Iz}(M)$ denote the set of points with nontrivial isotropy group (for a given G-action on M).

The following theorem contains the main result of our paper:

1.4. THEOREM. *Actions of Z_n on $S^1 \hat{\times} F$ with* $\dim \mathrm{Iz}(S^1 \hat{\times} F) \leqslant 0$ *are standard for $F = S^1, S^2, P^2, D^1, D^2$ except for the cases of Z_3-actions on $S^1 \times S^1$ with 3 fixed points and Z_4-actions on $S^1 \times S^1$ with 2 fixed points.*

2. Construction of standard actions. We have introduced in [11] some methods of constructing Z_n-actions. We will show that the standard actions (Definition 1.1) can be obtained as simple examples of those constructions.

2.1. Definition. Let n, r be a pair of natural numbers such that (i) $r \leqslant n$ and (ii) $(r, n) = 1$. Then we define $n(r)$ as a natural number which satisfies the following conditions:

$$n(r) \leqslant n, \qquad \underset{0 \leqslant k < n}{\exists} n(r)r = 1 + kn.$$

The function $n(\cdot)$ is a self-bijection of the set of numbers r satisfying conditions (i) and (ii). $k(n, r)$ is a well-defined function on pairs (n, r) which satisfy (i) and (ii).

Now we use the terminology of [11].

2.2. LEMMA. (a) $(1; n, s, i, g_0)_{(F,\varphi)} = ([0,1] \times F, \mathrm{Id} \times g)^s_{(\{0\} \times F, \{1\} \times F, f, s(i))}$, *where* $\mathrm{Id} \times g$ *generates a* $Z_{n/s}$*-action on* $[0,1] \times F$ *and* $g = \varphi^{-i} g_0^s$, $f = \varphi^{-k(s,i)} g_0^{s(i)}$.

Conversely, $([0,1] \times F, \mathrm{Id} \times g)^s_{(\{0\} \times F, \{1\} \times F, f, r)}$, *where* $\mathrm{Id} \times g$ *generates a* Z_{n_1}*-action on* $[0,1] \times F$, *is equal to* $(1; n_1 s, s, s(r), g_0)_{(F,\varphi)}$, *where* $g_0 = f^{s(r)} g^{-k(s,r)}$ *and* $\varphi = f^s g^{-r}$.

(b) $(2; n, g_0)_{(F,\varphi)} = ([0,1] \times F, g_1) \hat{\#}_{(F_1, F_2, f)} ([0,1] \times F, g_2)$, *where* $F_i = \{0\} \times F$ $(i = 1, 2)$, $g_1(t, y) = \big(1-t, g_0(y)\big)$, $g_2(t, y) = \big(1-t, f g_0 \varphi f^{-1}(y)\big)$, *and f is an arbitrary homeomorphism from F_1 to F_2.*

Conversely, $([0,1] \times F, g_1) \hat{\#}_{(\{0\} \times F, \{0\} \times F, f)} ([0,1] \times F, g_2)$, *where g_1 and g_2 generate Z_n-actions on* $[0,1] \times F$, $g_1(t, y) = \big(1-t, g_0'(y)\big)$ *and* $g_2(t, y) = \big(1-t, g_0''(y)\big)$ *for given self-homeomorphisms g_0' and g_0'' of F, is equal to* $(2; n, g_0')_{(F,\varphi)}$, *where* $\varphi = g_0' f^{-1} g_0''^{-1} f$.

Proof. The embedding $\{0\} \times F \hookrightarrow M$ satisfies the assumptions of Theorem 4.1 in [11]. This theorem reduces the proof of our lemma to a simple computation.

3. Actions on $S^1 \hat{\times} S^2$. Let $\mathscr{A}$ denote a class of actions. Then $\xi_c(\mathscr{A})$ (respectively, $\xi_w(\mathscr{A})$) denotes the number of actions in $\mathscr{A}$ up to conjugation (respectively, up to weak conjugation). Let $\varphi(n)$ denote the cardinality of the set of all natural numbers relatively prime to n, less than n, and let $[n]$ denote the integer part of n.

The following theorem extends the results contained in [4], [8], [14], [16], and [17].

3.1. THEOREM. *Each action of Z_n on $S^1 \hat{\times} S^2$ (generated by T) with* $\dim \mathrm{Iz}(S^1 \hat{\times} S^2) \leqslant 0$ *takes one of the following forms (up to conjugation). Each of the subcases* I.1(a), ..., II(c) *describes exactly one class of weakly conjugate actions.*

I. *Actions of Z_n on $M = S^1 \times S^2$ ($\varphi = \mathrm{Id}$).*

1. *Actions which preserve orientation:*

(a) *Actions of type* $(1;\, n, n, i, \mathrm{Id})$ (*i.e.,* $T(t, y) = (t+i/n, y)$). *Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c((\mathrm{a})) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = M_\varphi/Z_n = S^1 \times S^2,$$

where, for brevity, the set of actions described in I.1(a) *is denoted by* (a).

(b) *Action of type* $(2;\, 2, A)$ (i.e., $T(t, y) = (1-t, A)$), *where A denotes the antipodal map*; $\mathrm{Iz}(M_\varphi) = \emptyset$, *and* $M^* = P^3 \# P^3$.

2. *Actions which reverse orientation:*

(a) *Actions of type* $(1;\, n, n, i, A)$, *where n is even. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c((\mathrm{a})) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = N.$$

(b) *Actions of type* $(1;\, n, s, i, A)$, *where $n = 2s$, s and i are odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$; hence*

$$\xi_c((\mathrm{b})) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = S^1 \times P^2.$$

(c) *Action of type* $(2;\, 2, R)$, *where R is an involution of S^2 with 2 fixed points* ($S^2 = \{y_1, y_2, y_3 : y_1^2+y_2^2+y_3^2 = 1\}$, $R(y_1, y_2, y_3) = (-y_1, -y_2, y_3)$), $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = 4$ *points.*

II. *Actions of Z_n on $S^1 \hat{\times} S^2 = N$:*

(a) *Actions of type* $(1;\, n, n, i, A^i)$ *on M_A, where n is odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c((\mathrm{a})) = \frac{\varphi(n)}{2}, \quad \mathrm{Iz}(M_A) = \emptyset, \quad M^* = N.$$

(b) *Actions of type* $(1;\ n, s, i, \mathrm{Id})$ *on* $N = M_A$, *where* $n = 2s$, *and* i *is odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_A) = \emptyset, \quad M^* = S^1 \times P^2.$$

(c) *Action of type* $(2;\ 2, R)$ *on* $N = M_C$, *where* C *is an involution of* S^2 *with* $\mathrm{Fix}(C) = S^1$ $\big(C(y_1, y_2, y_3) = (y_1, y_2, -y_3)\big)$, $\mathrm{Iz}(M_C) = \mathrm{Fix}(T) = 2$ *points*.

For the proof we need two lemmas. Recall that two 3-manifolds M_1 and M_2 are said to be *congruent* if there exist two homotopy spheres Σ_1^3 and Σ_2^3 such that $M_1 \# \Sigma_1^3 = M_2 \# \Sigma_2^3$.

3.2. Lemma. *Let* T *be a generator of a* Z_n*-action on a* 3*-manifold* M *with* $\dim \mathrm{Iz}(M) \leqslant 0$. *If* M *is not congruent to an irreducible manifold, then there is an embedding* $S^2 \hookrightarrow \mathrm{int} M$, *bounding no* 3*-cell, such that for each* i *either* $S^2 \cap T^i(S^2) = \emptyset$ *or* $S^2 = T^i(S^2)$.

Lemma 3.2 is a consequence of Theorem 5.3 in [11].

3.3. Lemma. *An action of type* $(1;\ n, s, i, g_0)$ *on* $M_\varphi = S^1 \hat{\times} F$ *is conjugate to an action of type* $(1;\ n, s, s-i, \varphi a g_0 a^{-1})$, *where* a *is a self-homeomorphism of* F *which reverses orientation if* M *is orientable, and* $a\varphi a^{-1} = \varphi^{-1}$.

Indeed, the actions are conjugate by the homeomorphism given by $(t, y) \to \big(-t, a(y)\big)$.

Proof of Theorem 3.1. By Lemma 3.2, we may use Theorem 4.1 of [11] (for $F = S^2$) and Corollary 4.4 from [11] for actions on $S^1 \hat{\times} S^2$. We have the following possibilities:

1. The action of Z_n was obtained as a multiple. We deduce from the formulas in Proposition 3.5 of [11] that

$$(S^1 \hat{\times} S^2, Z_n) = ([0,1] \times S^2, g)^s_{(\{0\}\times S^2, \{1\}\times S^2, f, r)} \quad \text{and} \quad j_0 = 1$$

(see Definition 3.2 in [11]). The only free action on S^2 is the antipodal map. Hence $j = 1$ or 2 (see [11], 3.2). By Lemma 3.3 we have the following possibilities:

(a) $j = 1$, $g = \mathrm{Id}$, $s = n$ and either (i) or (ii) holds:

(i) $f = \mathrm{Id}$; then by Lemma 2.2(a) we obtain the actions described in Theorem 3.1, I.1(a);

(ii) $f = A$; then by Lemma 2.2(a) we obtain the actions described in Theorem 3.1, I.2(a), if n is even, and in Theorem 3.1, II(a) if n is odd.

(b) $j = 2$, $g = \mathrm{Id} \times A$ (we use the classification of involutions of S^3 given in [10]), $s = n/2$ and either (i) or (ii) holds:

(i) $f = \mathrm{Id}$; then by Lemmas 2.2(a) and 3.3 we obtain the actions described in Theorem 3.1, I.2(b), if r is even, and in Theorem 3.1, II(b), if r is odd;

(ii) $f = A$; then by Lemmas 2.2(a) and 3.3 we obtain the actions described in Theorem 3.1, I.2 (b), if $r+s$ is even and in Theorem 3.1, II(b), if $r+s$ is odd.

2. The action of Z_n was obtained by using a connected sum. We deduce from the formulas in Proposition 3.5 of [11] that

$$(S^1 \hat{\times} S^2, Z_n) = ([0,1]\times S^2, T_1) \,\hat{\#}_{(\{0\}\times S^2, \{0\}\times S^2, f)}([0,1]\times S^2, T_2)$$

and

$$s_1 = s_2 = 1, \quad j_0 = 2, \quad j = 1 \text{ or } 2$$

(see Definition 3.1 in [11]). Suppose that $j = 2$; then there is a Z_n-action on $[0,1]\times S^2$ (generated by a) such that $a^2|\{0\}\times S^2$ is the antipodal map. Thus a^2 reverses orientation of $[0,1]\times S^2$, which is not possible. Thus $j = 1$ and g_1, g_2 are involutions of $[0,1]\times S^2$. By [9] and [10], there exist exactly two involutions of $[0,1]\times S^2$, namely, $A_0\times A$ and $A_0\times R$, where $A_0(T) = 1-t$, which change boundary components of $[0,1]\times S^2$ and which satisfy $\dim \mathrm{Iz}([0,1]\times S^2)\leqslant 0$. Thus, we have the following possibilities (see Proposition 3.7 in [11]):

(a) $g_N = g_M = A_0\times A$; then by Lemma 2.2(b) we obtain the action described in Theorem 3.1, I.1(b);

(b) $g_M = A_0\times A$, $g_N = A_0\times R$; then we obtain the action described in Theorem 3.1, II(c);

(c) $g_M = g_N = A_0\times R$; then we obtain the action described in Theorem 3.1, I.2(c).

Actions described in distinct subcases I.1(a), ..., II(c) of Theorem 3.1 are not weakly conjugate because either the corresponding spaces of actions or orbit spaces, or the sets $\mathrm{Iz}(\cdot)$ are different. The classification of actions, described in each of the subcases of Theorem 3.1, is completed by Lemma 3.3 and by the following

3.4. PROPOSITION. *Let $\mathscr{A}$ denote the totality of free Z_n-actions whose orbit spaces are homeomorphic to M^*. Then there is a one-to-one correspondence between elements of $\mathscr{A}$, up to conjugation (respectively, weak conjugation), and equivalence classes (respectively, weak equivalence classes) of epimorphisms $H_1(M^*, Z)\to Z_n$.*

Recall that two epimorphisms a_1 and a_2 are said to be *weakly equivalent* if there exist a group automorphism $A\colon Z_n\to Z_n$ and a self-homeomorphism $f\colon M^*\to M^*$, which preserves orientation if M^* is orientable, such that $Aa_1 = a_2 f_*$. If A is the identity, then a_1 and a_2 are equivalent.

For the proof of Proposition 3.4 see, e.g., [15].

Thus the proof of Theorem 3.1 is complete.

4. Actions on $S^1 \times P^2$.

4.1. THEOREM. *Each action of Z_n on $M_\varphi = S^1 \times P^2$ ($\varphi = \mathrm{Id}$) with $\dim \mathrm{Iz}(M) \leqslant 0$ is of type $(1;\ n, n, i, \mathrm{Id})$. Two such actions, for $i = i', i''$, are always weakly conjugate, and they are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c(\cdot) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = S^1 \times P^2.$$

Proof. We deduce from Corollary 5.6 in [11] that there exists a 2-sided embedding $P^2 \hookrightarrow S^1 \times P^2$ such that $T^i(P^2) \cap P^2 = \emptyset$ for each i $(0 < i < n)$. Now, the method of proof is similar to that of Theorem 3.1. We use Corollary 4.4 from [11] instead of Corollary 4.3 from [11]. Note that $I \times P^2$ and P^2 do not admit involutions with $\dim \mathrm{Fix}(\cdot) \leqslant 0$.

5. Actions on $S^1 \hat{\times} D^1$.

5.1. THEOREM. *Each effective action of Z_n on $S^1 \hat{\times} D^1$ (generated by T) takes one of the following forms (up to conjugation). Each of the subcases* I.1(a), ..., II(c) *describes exactly one class of weakly conjugate actions.*

I. *Actions on $M_\varphi = S^1 \times D^1$ ($\varphi = \mathrm{Id}$).*

1. *Actions which preserve orientation:*

(a) *Actions of type $(1;\ n, n, i, \mathrm{Id})$. Two such actions, for $i = i', ii''$, are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = S^1 \times D^1.$$

(b) *Action of type $(2;\ 2, A)$, where A is the antipodal map;* $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = 2$ *points.*

2. *Actions which reverse orientation:*

(a) *Actions of type $(1;\ n, n, i, A)$, where n is even. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $n-i'$; hence*

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = Mb.$$

(b) *Actions of type $(1; n, s, i, A)$, where $n = 2s$, s and i are odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$; hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T^s) = S^1, \quad M^* = S^1 \times D^1.$$

(c) *Actions of type $(2;\ 2, \mathrm{Id})$;* $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = D^1 \mathring{\cup} D^1$, $M^* = D^1 \times D^1$.

II. *Actions on* $M_\varphi = S^1 \hat{\times} D^1 = Mb$ $(\varphi = A)$:

(a) *Actions of type* $(1;\ n, n, i, A)$, *where* n *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{a})\big) = \frac{\varphi(n)}{2}, \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = Mb.$$

(b) *Actions of type* $(1;\ n, s, i, \mathrm{Id})$, *where* $n = 2s$, *and* i *is odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = S^1, \quad M^* = S^1 \times D^1.$$

(c) *Action of type* $(2;\ 2, \mathrm{Id})$; $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = D^1 \cup$ *point*, $M^* = D^2$.

Proof. Theorem 5.1 is probably well-known. We prove it in a similar manner as Theorem 3.1 by using Theorem 4.1 from [11] for $F = D^1$. We also use the fact that for each effective action of any finite group G on $S^1 \hat{\times} D^1$ there exists a properly embedded 1-disk D^1 which does not cut out a 2-disk and such that $g(D^1) \cap D^1 = \emptyset$ or D^1 for each $g \in G$.

Theorem 5.1 enables us to classify easily the effective actions of finite groups on $S^1 \hat{\times} D^1$.

5.2. Definition. (a) Let

$$G_1 = Z_2 \oplus D_{2n} = \{a, b, c\colon\ a^2, b^n, c^2, aca^{-1}c^{-1}, bcb^{-1}c^{-1}, abab\}.$$

G_1 acts effectively on $M_\varphi = S^1 \times D^1$ $(\varphi = \mathrm{Id})$ and T_a, T_b, T_c are self-homeomorphisms of M_φ which correspond to the generators of G_1:

$$T_a(t, y) = (1-t, y), \quad T_b(t, y) = (t+1/n, y), \quad T_c(t, y) = \big(t, A(y)\big).$$

(b) Let

$$G_2 = D_{2n} = \{a, b\colon\ a^2, b^n, abab\}.$$

G_2 acts effectively on $Mb = M_\varphi = S^1 \hat{\times} D^1$ $(\varphi = A)$ and T_a, T_b are self-homeomorphisms of M_φ which correspond to the generators of G_2:

$$T_a(t, y) = (1-t, y), \quad T_b(t, y) = (t+2/n, y).$$

5.3. THEOREM. (a) *Each effective action of a finite group on* $S^1 \times D^1$ *is conjugate to an action of some subgroup of* G_1.

(b) *Each effective action of a finite group on* Mb *is conjugate to an action of some subgroup of* G_2.

6. Actions on $S^1 \hat{\times} D^2$.

6.1. Definition. We say that an n-manifold M with an action of a finite group G satisfies *condition* $(*)$ if there exists a compact $(n-1)$-manifold F such that $M = S^1 \hat{\times} F$ and there is a properly embedded, 2-sided $F \hookrightarrow S^1 \hat{\times} F$ which satisfies the following conditions:

(i) $M - F$ is connected,

(ii) F is in a general position with respect to $\mathrm{Iz}(M)$,

(iii) for each $g \in G$, $g(F) \cap F = \emptyset$ or $g(F) = F$.

6.2. LEMMA. (a) *$S^1 \hat{\times} D^2$ with an action of a finite group G, where* $\dim \mathrm{Iz}(S^1 \hat{\times} D^2) \leqslant 0$, *satisfies condition* $(*)$.

(b) *Each involution on $S^1 \hat{\times} D^2$ satisfies condition* $(*)$.

(c) *$S^1 \hat{\times} D^2$ with a Z_{2^n}-action, where*

$$\exists_{1 \leqslant j < 2^n} \operatorname{Fix}(T^j) = S^1,$$

satisfies condition $(*)$.

Proof. (a) follows from Theorem 5.7 in [11], and (b) follows from Lemma 2 in [7]. For the proof of (c) see Lemma 2.8 in [5].

Let $D^2 = \{z \in \text{complex numbers}: |z| \leqslant 1\}$. Now we introduce the following self-homeomorphisms of D^2:

$$C(z) = \bar{z}, \quad A(z) = -z, \quad O_a(z) = e^{ia}z, \quad g_{(n,h)} = O_{2\pi h/n},$$

where h is taken modulo n.

6.3. CONJECTURE (P. A. Smith). Each orientation-preserving Z_n-action on $D^3 = [0, 1] \times D^2$ is conjugate to an orthogonal action (i.e., $(t, z) \to (t, g_{(n,h)})$ for some h).

6.4. Remark (Waldhausen [19]). The Smith Conjecture is true for even n.

The following theorem extends the results of Kim [5]:

6.5. THEOREM. *Each effective action of Z_n on $S^1 \hat{\times} D^2$ (generated by T), which satisfies condition $(*)$, takes one of the following forms (up to conjugation). Each of the subcases* I.1(a$'$), ..., II(d) *describes exactly one class of weakly conjugate actions. We assume additionally that the Smith Conjecture* (6.3) *is true for each k which divides n.*

I. *Actions on $S^1 \times D^2$.*

1. *Actions which preserve orientation* (*on* $M_\varphi = S^1 \times D^2$, $\varphi = \mathrm{Id}$):

(a) *Actions of type* $(1; n, s, i, g_{(n,h)})$, *where* $j = n/s$, $(j, h) = 1$, $0 < h \leqslant j$. *Two such actions, for* $s = s', s''$, $h = h', h''$, $i = i', i''$, *are weakly conjugate iff* $s'' = s'$ *and there exists a natural number a such that either*

$$i'' \equiv ai' \pmod{s'} \quad \textit{and} \quad h'' \equiv ah' \pmod{j'}$$

or

$$i'' \equiv a(s'-i') \pmod{s'} \quad \textit{and} \quad h'' \equiv a(j'-h') \pmod{j'}.$$

If $a = 1$, then the actions are conjugate. For given s we have

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(s)\varphi(j)+1}{2}\right], \quad \xi_w\big((\mathrm{a})\big) = \varphi\big(\text{g.c.d.}\,(s,j)\big).$$

Each class of the actions (up to weak conjugation) contains exactly $\left[2^{-1}\big(\varphi\big(\text{s.c.m.}(s,j)\big)+1\big)\right]$ *actions, up to conjugation.* $M^* = S^1 \times D^2$; *if* $j > 0$, *then* $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T^s) = S^1$. *In particular, if* $j = 1$, *then we obtain*:

(a′) *Actions of type* $(1;\, n, n, i, \mathrm{Id})$. *Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $n - i'$; *hence*

$$\xi_c\big((\mathrm{a}')\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = S^1 \times D^2.$$

(b) *Action of type* $(2;\, 2, C)$, $\mathrm{Iz}(M_\varphi) = D^1 \overset{\bullet}{\cup} D^1$, $M^* = D^3$.

2. *Actions which reverse orientation*:

(a) *Actions of type* $(1;\, n, n, i, C)$ *on* $M_\varphi = S^1 \times D^2$ $(\varphi = \mathrm{Id})$, *where* n *is even. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $n - i'$; *hence*

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = Bs.$$

(b) *Actions of type* $(1;\, n, s, i, C)$ *on* M_{Id}, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_{\mathrm{Id}}) = \mathrm{Fix}(T^s) = S^1 \times D^1, \quad M^* = S^1 \times D^2.$$

(c) *Actions of type* $(1;\, n, s, i, AC)$ *on* M_A, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_A) = \mathrm{Fix}(T^s) = Mb, \quad M^* = Bs.$$

(d) *Actions of type* $(1;\, n, s, i, C)$ *on* M_A, *where* $n = 2s$, s *is even. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $s - i'$; *hence*

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(s)+1}{2}\right], \quad \mathrm{Iz}(M_A) = \mathrm{Fix}(T^s) = S^1, \quad M^* = Bs.$$

(e) *Actions of type* $(2;\, n, g_{(n,h)})$ *on* M_{Id}, *where* n *is even*, $(n, h) = 1$, $0 < h \leqslant n$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n - h'$; *hence*

$$\xi_c\big((\mathrm{e})\big) = \left[\frac{\varphi(n)+1}{2}\right],$$

$$\mathrm{Iz}(M_{\mathrm{Id}}) = \begin{cases} \mathrm{Fix}(T^2) = S^1 & \textit{if } n > 2, \\ 2 \textit{ points} & \textit{if } n = 2, \end{cases} \quad \mathrm{Fix}(T) = 2 \textit{ points}.$$

(f) *Actions of type* $(2;\, n, g_{(n,h)})$ *on* M_A, *where* n *is even. Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$, $n-h'$, $h' \mp n/2$ *or* $n - h' \mp$ $\mp n/2$; *hence*

$$\mathrm{Iz}(M_A) = \begin{cases} \mathrm{Fix}(T) = D^2 \cup \textit{point} & \textit{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \cup D^2 & \textit{if } n > 2, n/2 \textit{ is odd}, \\ \mathrm{Fix}(T^2) = S^1 & \textit{if } n/2 \textit{ is even}, \end{cases}$$

$$\mathrm{Fix}(T) = \begin{cases} D^2 \cup \textit{point} & \textit{if } n = 2, \\ 2 \textit{ points} & \textit{if } n > 2, \end{cases} \quad \xi_c\big((\mathrm{f})\big) = \left[\frac{\varphi(n/2)+1}{2}\right].$$

(g) *Actions of type* $(2;\, n, g_{(n/2,h)})$ *on* M_{Id}, *where* n *is even*, $n/2$ *and* h *are odd. Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$; *hence*

$$\xi_c\big((\mathrm{g})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} D^2 \overset{\circ}{\cup} D^2 & \textit{if } n = 2, \\ 2 \textit{ points} & \textit{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M_{\mathrm{Id}}) = \begin{cases} \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \cup (D^2 \overset{\circ}{\cup} D^2) & \textit{if } n > 2, \\ \mathrm{Fix}(T) = D^2 \overset{\circ}{\cup} D^2 & \textit{if } n = 2, \end{cases} \quad M^* = D^1 \times D^2.$$

II. *Actions on* $M_\varphi = S^1 \hat{\times} D^2 = Bs$ $(\varphi = C)$:

(a) *Actions of type* $(1;\, n, n, i, C)$, *where* n *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{a})\big) = \frac{\varphi(n)}{2}, \quad \mathrm{Iz}(M_\varphi) = \emptyset, \quad M^* = Bs.$$

(b) *Actions of type* $(1;\, n, s, i, AC)$, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T^s) = S^1, \quad M^* = Bs.$$

(c) *Actions of type* $(1;\ n, s, i, \mathrm{Id})$, *where* $n = 2s$, *and* i *is odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T^s) = S^1\times D^1, \quad M^* = S^1\times D^2.$$

(d) *Actions of type* $(1;\ n, s, i, A)$, *where* $n = 2s$, *and* i *is odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(n)+1}{2}\right],$$

$$\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T^s) = \begin{cases} S^1\times D^1 & \textit{if } s \textit{ is even},\\ Mb & \textit{if } s \textit{ is odd},\end{cases} \qquad M^* = Bs.$$

(e) *Action of type* $(2;\ 2, CA)$; $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = D^1 \cup 1$ *point*.

(f) *Action of type* $(2;\ 2, C)$; $\mathrm{Iz}(M_\varphi) = \mathrm{Fix}(T) = D^1 \mathring{\cup} D^2$, $M^* = D^3$.

Proof of Theorem 6.5 is similar to that of Theorem 3.1. We give only the outline. It follows from condition $(*)$ and Corollary 4.5 in [11] that each Z_n-action on $S^1 \hat{\times} D^2$ is obtained by using a multiple or a connected sum for $F = D^2$. We have the following possibilities (we use the terminology of 3.1 and 3.2 from [11]):

1. Case of multiple. $M_1 = [0,1]\times D^2$. We obtain an action on $S^1 \hat{\times} D^2$. Hence $j_0 = 1$. Let $F_i = D_i^2 = \{i-1\}\times D^2$ $(i = 1, 2)$.

(a) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = \emptyset$. Then $j = 1$, $n = s$, $g = \mathrm{Id}$. A simple computation (using Lemma 2.2 and Proposition 3.13 from [11]) leads us to the actions described in I.1(a′), I.2(a) or II(a) of Theorem 6.5.

(b) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = 1$ point. Then $j > 1$, $g|D_1^2 = g_{(j,h)}$. We deduce from the Smith Conjecture (6.3) that $g = \mathrm{Id}\times g_{(j,h)}$. Now, by a simple calculation (using Lemma 2.2) we obtain either the actions described in I.1(a) if $f\colon D_1^2 \to D_2^2$ reverses orientation (D_i^2 is oriented in agreement with the orientation of $[0,1]\times D^2$; we may assume that $0 < h \leqslant j$ by Fact 6.7 below) or the actions described in I.2(d) or II(b) if f preserves orientation.

(c) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = D^1$. Then $g|D_1^2 = C$. Hence $g = \mathrm{Id}\times C$ (for f we have 4 possibilities: $f = \mathrm{Id}, C, A$ or AC). Now, simple calculations (using Lemma 2.2) lead us to actions described in I.2(b), I.2(c), II(c) or II(d) (note that the actions of type $(1;\ n, s, i, A)$ on M_A and on M_{AC} are conjugate).

2. Case of connected sum. $M_1 = M_2 = [0,1]\times D^2$. We obtain an action on $S^1 \hat{\times} D^2$. Hence $j_0 = 2$ and $s_1 = s_2 = 1$. Let $F_i = D_i^2 = \{0\}\times D^2$ $(i = 1, 2)$.

(a) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = \emptyset$. Then $j = 1$. Since the only involutions on $[0, 1] \times D^2$ (free on the boundary) are $A_0 \times \mathrm{Id}$ and $A_0 \times C$, after simple computations (using Lemma 2.2(b) and Proposition 3.13 from [11]) we obtain one of the actions described in I.1(b), I.2(e), I.2(f) (for $n = 2$), II(e) or II(f) of Theorem 6.5.

(b) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = 1$ point. Then $T_i^2 | D_i^2$ $(i = 1, 2)$ is of type $g_{(j,h_i)}$, where $n = 2j$. We infer, using the Smith Conjecture and the result of Kim [6], that T_1 is of type $(2;\ n,\ g_{(n,h)})$, where n is even, or of type $(2;\ n,\ g_{(n/2,h)})$, where n is even and $n/2$ is odd (similarly T_2). Now, simple computations (which exhaust all the possibilities for f) and the use of Lemma 2.2(b) lead us to the actions described in I.2(e), (f), (g) of Theorem 6.5.

(c) $D_1^2 \cap \mathrm{Iz}(S^1 \hat{\times} D^2) = D^1$. This case cannot occur.

To complete the proof of Theorem 6.5 it remains to verify the following:

I. Actions described in distinct subcases I.1(a), I.1(b), ..., II(f) of Theorem 6.5 are not weakly conjugate. These actions differ either by the space of action or by the orbit space, or by the set $\mathrm{Iz}(\cdot)$, except for the case of the actions I.2(e) and I.2(f), where $n/2$ is even.

It is sufficient to consider the case $n = 4$ because otherwise we have different actions in the neighborhoods of fixed points. We distinguish these actions as follows: generators T_1 and T_2 of the actions I.2(e) and I.2(f), respectively, are defined by the formulas

$$T_1(t, z) = \big(1-t,\ g_{(4,1)}(z)\big) \quad \text{and} \quad T_2(t, z) = (1-t,\ O_{2\pi t-\pi/2}).$$

Both actions are on $M_{\mathrm{Id}} = S^1 \times D^2$. Let

$$\gamma = S^1 \times \{0\} = \mathrm{Fix}(T_1^2) = \mathrm{Fix}(T_2^2), \quad a = \{0\} \times \{0\}, \quad a' = \{\tfrac{1}{2}\} \times \{0\}$$

$(\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = a \cup a')$. Let a_1 and a_2 be self-homeomorphisms of M_{Id} defined by the formulas

$$a_1(t, z) = (t+\tfrac{1}{2},\ z) \quad \text{and} \quad a_2(t, z) = \big(\tfrac{1}{2}-t,\ C(z)\big).$$

a_i is T_i-equivariant and satisfies $a_i(a) = a'$, $a_i(a') = a$, $a_i(\gamma) = \gamma$ $(i = 1, 2)$. Furthermore, a_1 preserves the orientation of γ, and a_2 reverses the orientation of γ. We assume that there exists a self-homeomorphism β of M_{Id} which conjugates T_1 and T_2 (i.e., $\beta T_1 \beta^{-1} = T_2$). From the existence of a_1 and a_2 it follows that we may assume that (i) $\beta(a) = a$, $\beta(a') = a'$, (ii) $\beta(\gamma) = \gamma$ and β/γ preserves orientation. β maps a T_1-equivariant regular neighborhood of a' onto a T_2-equivariant regular neighborhood of a'. The following fact gives a contradiction:

6.6. FACT. *Let T_i' $(i = 1, 2)$ be a generator of a Z_n-action on $[0, 1] \times D^2$ defined by the formulas*

$$T_1'(t, z) = \big(1-t, g_{(4,1)}(z)\big) \quad \textit{and} \quad T_2'(t, z) = \big(1-t, g_{(4,3)}(z)\big).$$

Then each self-homeomorphism which conjugates T_1' and T_2' reverses the orientation on $[0, 1] \times \{0\} = \mathrm{Fix}(T_1'^2) = \mathrm{Fix}(T_2'^2)$.

II. We have to classify the actions described in all subcases of Theorem 6.5. First, we consider the actions described in the subcase I.1(a).

6.7. FACT. *Actions given by*

$$T_1(t, z) = \big(t+i/s, g_{(n,h)}(z)\big) \quad \textit{and} \quad T_2(t, z) = \big(t+i/s, g_{(n,h+j)}(z)\big)$$

on $M_{\mathrm{Id}} = S^1 \times D^2$ are conjugate $(n = js)$.

Proof. We conjugate these actions using the function $(t, z) \to \to \big(t, O_{2\pi t}(z)\big)$.

By Lemma 3.3 and Fact 6.7 we know when the actions I.1(a) are conjugate (weak conjugate). We can distinguish these actions by comparing the action of T on $\mathrm{Iz}(\cdot) = S^1$ and the action of T^s on $S^1 \times D^2$.

We classify the actions in the remaining cases of Theorem 6.5 using Lemma 3.3 and Proposition 3.4 (for M_φ, $M_\varphi - \mathrm{Iz}(\cdot)$, $\mathrm{Iz}(\cdot)$ or M_φ/T^j for some j). This completes the proof of Theorem 6.5.

7. Actions on $S^1 \hat{\times} S^1$. Let A, C, O_a, $g_{(n,h)}$ be self-homeomorphisms of S^1 which are the cuts to the boundary of the corresponding self-homeomorphisms on D^2.

7.1. THEOREM. *Each effective action of Z_n on $S^1 \hat{\times} S^1$ (generated by T) takes one of the following forms (up to conjugation). Each of the subcases* I.1(a), ..., II(g) *describes exactly one class of weakly conjugate actions.*

I. *Actions on $S^1 \times S^1$.*

1. *Actions which preserve orientation (on $M_{\mathrm{Id}} = S^1 \times S^1$):*

(a) *Action of type* $(1;\ n, n, 1, \mathrm{Id})$; $\mathrm{Iz}(M_{\mathrm{Id}}) = \emptyset$, $M^* = S^1 \times S^1$.

(b) *Action of type* $(2;\ 2, C)$; $\mathrm{Iz}(M_{\mathrm{Id}}) = 4$ *points*, $M^* = S^2$.

(c) *Nonstandard actions of Z_3 on $S^1 \times S^1$. These actions (two up to conjugation) are constructed as follows:*

We use a connected sum (terminology of 3.1 *in* [11]). *Let*

$$\big(S^1 \times S^1 \# (D^2)_3, T'\big) = (1;\ 3, 3, 1, \mathrm{Id})_{(D^1,\mathrm{Id})} \overline{\#}_{(D_1^1, D_2^1, f)} (1; 3, 3, 1, \mathrm{Id})_{(D^1,\mathrm{Id})}$$

and

$$j = s_1 = s_2 = 1, \quad j_0 = 3, \quad D_1^1, D_2^1 \subset S^1 \times \{1\}.$$

Now, we extend this action to $S^1 \times S^1$. If T is a generator of this action, then T^2 generates the second action in (c). $\mathrm{Iz}(S^1 \times S^1) = 3$ *points*, $M^* = S^2$.

(d) *Nonstandard actions of* Z_4 *on* $S^1 \times S^1$. *These actions* (*two up to conjugation*) *are constructed as follows:*

We use a connected sum. Let

$$\left(S^1 \times S^1 \# (D^2)_4, T'\right) = (1; 4, 4, 1, \mathrm{Id})_{(D^1, \mathrm{Id})} \overline{\#}_{(D_1^1, D_2^1, f)} (1; 4, 4, 1, \mathrm{Id})_{(D^1, \mathrm{Id})}$$

and

$$j = s_1 = s_2 = 1, \quad j_0 = 4, \quad D_1^1, D_2^1 \subset S^1 \times \{1\}.$$

Now, we extend this action to $S^1 \times S^1$. *If* T *is a generator of this action, then* T^3 *generates the second action in* (d). $\mathrm{Iz}(S^1 \times S^1) = 4$ *points*, $\mathrm{Fix}(T) = 2$ *points*, $M^* = S^2$.

2. *Actions which reverse orientation:*

(a) *Actions of type* $(1; n, n, i, C)$ *on* $M_{\mathrm{Id}} = S^1 \times S^1$, *where* n *is even. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $n - i'$; *hence*

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_{\mathrm{Id}}) = \emptyset, \quad M^* = B.$$

(b) *Actions of type* $(1; n, s, i, C)$ *on* M_A, *where* $n = 2s$ *and* s *is even. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $s - i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(s)+1}{2}\right], \quad \mathrm{Iz}(M_A) = \emptyset, \quad M^* = B.$$

(c) *Actions of type* $(2; n, g_{(s,i)})$ *on* M_{Id}, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_{\mathrm{Id}}) = \mathrm{Fix}(T^s) = S^1 \mathring{\cup} S^1, \quad M^* = S^1 \times D^1.$$

(d) *Actions of type* $(2; n, g_{(s,i)})$ *on* M_A, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_A) = \mathrm{Fix}(T^s) = S^1, \quad M^* = Mb.$$

II. *Actions on* $M_C = S^1 \hat{\times} S^1 = B$:

(a) *Actions of type* $(1; n, n, i, C^i)$, *where* n *is odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $n - i'$; *hence*

$$\xi_c\big((\mathrm{a})\big) = \frac{\varphi(n)}{2}, \quad \mathrm{Iz}(M_C) = \emptyset, \quad M^* = B.$$

(b) *Actions of type* $(1; n, s, i, AC)$, *where* $n = 2s$, s *and* i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M_C) = \emptyset, \quad M^* = B.$$

(c) *Action of type* $(2;\ 2, A)$; $\mathrm{Iz}(M_C) = \mathrm{Fix}(T) = 2$ *points*, $M^* = P^2$.

(d) *Action of type* $(2;\ 2, \mathrm{Id})$; $\mathrm{Iz}(M_C) = \mathrm{Fix}(T) = S^1 \cup 2$ *points*, $M^* = D^2$.

(e) *Nonstandard actions obtained by using a connected sum*

$$(B, T) = (M_1, T_1)\ \hat{\#}_{(F_1, F_2, f)}(M_2, T_2),$$

where $(M_1, T_1) = (1;\ n, n/2, i, A)_{(D^1, \mathrm{Id})}$, $j = n/2$, $j_0 = s_1 = 1$, $s_2 = 2$, $n/2$ *and* i *are odd*, $M_2 = Mb$, $T_2(t, y) = (t + 4i/n, y)$, $F_1 = S^1 \times \{0\} \subset \partial M_1$, *and* $F_2 = \partial M_2$. *Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{e})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(B) = \mathrm{Fix}(T^{n/2}) = S^1, \quad M^* = Mb.$$

(f) *Nonstandard actions obtained by using a connected sum*

$$(B, T) = (M_1, T_1)\ \hat{\#}_{(F_1, F_2, f)}(M_2, T_2),$$

where $(M_1, T_1) = (1;\ n, n, i, A)_{(D^1, \mathrm{Id})}$, n *is a multiple of* 4, $j = n/2$, $j_0 = s_1 = 1$, $s_2 = 2$, $M_2 = Mb$, $T_2(t, y) = (t + 4i/n, y)$, $F_1 = S^1 \times \{0\} \subset \partial M_1$, *and* $F_2 = \partial M_2$. *Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{f})\big) = \frac{\varphi(n)}{2}, \quad \mathrm{Iz}(B) = \mathrm{Fix}(T^{n/2}) = S^1 \mathring{\cup} S^1, \quad M^* = Mb.$$

(g) *Nonstandard actions obtained by using a connected sum*

$$(B, T) = (Mb, T_1)\ \hat{\#}_{(\partial Mb, \partial Mb, f)}(Mb, T_2),$$

where $(Mb, T_1) = (Mb, T_2) = (1;\ n, s, i, \mathrm{Id})_{(D^1, A)}$ *with* $n = 2s$ *and* i *odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{g})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(B) = \mathrm{Fix}(T^s) = S^1 \mathring{\cup} S^1, \quad M^* = S^1 \times D^1.$$

Outline of the proof. The method of proof is similar to that of Theorems 3.1, 4.1, 5.1, and 6.5. The difference is that we do not always find a circle in $S^1 \hat{\times} S^1$ which satisfies condition $(*)$ (Definition 6.1).

1. Free actions on $M = S^1 \hat{\times} S^1$.

(a) The case of a free orientation-preserving action on $S^1 \hat{\times} S^1$ is described, e.g., in Theorem 2 from [12].

(b) We have $M^* = B$ in all other cases. The classification of Z_n-actions on $M = B$ and orientation-reversing actions on $M = S^1 \times S^1$ is reduced (by using Proposition 3.4) to the classification of epimorphisms $\varphi: H_1(B, Z) \to Z_n$. We obtain this classification as follows:

Let (a, b) be standard generators of $\pi_1(B)$. Then

$$H_1(B,Z) = \pi_1(B)/[\pi_1(B), \pi_1(B)] = \{a, b : a^2b^2\}/[\pi_1(B), \pi_1(B)]$$
$$= \{a, b : a^2b^2, aba^{-1}b^{-1}\} = Z \oplus Z_2.$$

$H_0 \overset{\mathrm{df}}{=} \ker\varphi$ can take one of the following forms (up to equivalence):

(i) H_0 is generated by a^n and ab. Then $M = B$ if n is odd, and $M = S^1 \times S^1$ if n is even. We obtain the actions described in II(a) and I.2(a) of Theorem 7.1, respectively.

(ii) $n = 2s$; H_0 is generated by $b^{s+1}a$. Then $M = B$ if s is odd, and $M = S^1 \times S^1$ if s is even. We obtain the actions described in II(b) and I.2(b), respectively.

(ii′) $n = 2s$; H_0 is generated by $a^{s+1}b$. This case is the same as case (ii) because there exists a self-homeomorphism of B which changes the generators a and b.

H_0 determines the action up to weak conjugation. We obtain the classification (up to conjugation) using Proposition 3.4 and the fact that each self-homeomorphism of B can map a onto a, $-a$, b or $-b$ exclusively.

2. Action on $M = S^1 \hat{\times} S^1$ with $\dim \mathrm{Iz}(M) = 0$.

Let $p: M \to M^*$, $p(\mathrm{Iz}(M)) = m$ points $a_1, a_2, \ldots, a_m$, where $m > 0$, $p|M - \mathrm{Iz}(M)$ is a covering. For each i let n/m_i denote the cardinality of the set $p^{-1}(a_i)$. Then

$$n(\chi(M^*) - m) = \chi(S^1 \hat{\times} S^1) - \sum_{i=1}^{m} n/m_i = -\sum_{i=1}^{m} n/m_i.$$

Hence $m - \chi(M^*) = \sum_{i=1}^{m} 1/m_i$; of course, $n \geqslant m_i > 1$ for each i, and the cardinality of $\mathrm{Iz}(M)$ is equal to $\sum_{i=1}^{m} n/m_i$. Now we may deduce that $\chi(M^*) = 2$ or 1, and so $M^* = S^2$ or P^2. Furthermore, it follows from Smith's theory [1] that:

7.2. *The cardinality of* $\mathrm{Fix}(T^i)$ *is less than or equal to* 4 *for* $0 < i < n$.

We have the following possibilities:

(a) $M^* = S^2$; then

$$m - 2 = \sum_{i=1}^{m} 1/m_i,$$

which gives us 3 possibilities:

(i) $m = 4$; $4-2 = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ ($n = 2$ by 7.2). We consider a circle $S_0^1 \hookrightarrow M^*$ which separates the points a_1 and a_2 from a_3 and a_4. We lift this circle to M and use Theorem 4.1 from [11] for $F = S^1$. Since S_0^1 disconnects M, case (a) (i) is obtained by using the connected sum (the terminology of 3.1 in [11]) for $s_1 = s_2 = 1$, $j_0 = 2$ (if $j_0 \neq 2$, then there exists an action on S^2 with 3 fixed points, which is not possible), $T_1 = T_2$ act on $M_1 = M_2 = S^1 \times [0,1]$ ($T_1(t, y) = (1-t, 1-y)$), and $F_i = S^1 \times \{0\} \hookrightarrow M_i$ ($i = 1, 2$). Thus we obtain the action described in I.1(b) of Theorem 7.1.

(ii) $m = 3$; $3-2 = \frac{1}{4}+\frac{1}{4}+\frac{1}{2}$ ($n = 4$ by 7.2). We consider a small, Z_4-invariant neighborhood V_{Iz} of $\mathrm{Iz}(M)$ in M. Let $\hat{p} = p|M - \mathrm{int}(V_{\mathrm{Iz}})$. $\mathrm{Im}(\hat{p})$ is a 2-sphere with 3 holes. The covering of the boundary, say ∂_1, of some hole consists of two 1-spheres. Let D_1 be a 1-disk which disconnects $\mathrm{Im}(\hat{p})$ into 2 annuli and let the ends of D_1 be in ∂_1. We lift D_1 to $M - \mathrm{int}(V_{\mathrm{Iz}})$. Since D_1 disconnects $\mathrm{Im}(\hat{p})$, we have to do with a connected sum for $F_1, F_2 = D^1$ exactly in the same manner as in I.1(d) of Theorem 7.1 (up to conjugation).

(iii) $m = 3$; $3-2 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ ($n = 3$ by 7.2). We may deduce, similarly as in (ii), that the actions considered in (iii) are the same as the actions described in I.1(c) of Theorem 7.1.

(b) $M^* = P^2$; then

$$m-1 = \sum_{i=1}^{m} 1/m_i.$$

Hence $m = 2$, $2-1 = \frac{1}{2}+\frac{1}{2}$ ($n = 2$ or 4 by 7.2). We may deduce, similarly as in (a) (i) that $n = 2$ and we obtain the action described in II(c) of Theorem 7.1.

I.1(c) (and I.1(d)) contains two actions (up to conjugation) because each orientation-preserving homeomorphism of $S^1 \times S^1 \# (D^2)_i$ maps boundary onto boundary and preserves orientation of the boundary (we consider actions on $M - \mathrm{int}(X)$, where X is a regular neighborhood of $\mathrm{Iz}(M)$). We use Proposition 3.4.

3. Actions on $M = S^1 \hat{\times} S^1$ with $\dim \mathrm{Iz}(M) = 1$. Of course, n is even: $n = 2s$. If $x \in \mathrm{Iz}(M)$, then $T^i(x) \in \mathrm{Iz}(M)$ for each i and $\dim \mathrm{Fix}(T^s) = 1$. We have two possibilities (a) and (b):

(a) There is a 2-sided embedding $S^1 \hookrightarrow M$ such that $S^1 \hookrightarrow \mathrm{Fix}(T^s)$.

We use the following lemma:

7.3. LEMMA. *If S^1 is 2-sided in M and $S^1 \hookrightarrow \mathrm{Iz}(M)$, then S^1 is Z_n-invariant (i.e., $T^i(S^1) = S^1$ for each i).*

Proof. By Smith's theory [1], $\mathrm{Iz}(M)$ contains at most 2 circles. Assume that $S_0^1 = T(S^1) \neq S^1$. If $M = S^1 \times S^1$, then T^s reverses orientation. Thus s is odd, which contradicts the fact that $T^s(S^1) = S^1$. If $M = B$, then $B - S_0^1 - S^1$ is not connected and has exactly two components (T^s changes these components). Thus s is odd, which contradicts the fact that $T^s(S^1) = S^1$.

Now, we know that $T(S^1) = S^1$ in (a). Let M_1 be a small, T-invariant, regular neighborhood of S^1. By Theorem 4.1 in [11], (M, T) is obtained by using a connected sum for $F_1 = F_2$ being a circle, $j = n/2$, $M_1 = S^1 \times D^1$ with an action of type $(1;\ n, n/2, i, A)_{(D^1, \mathrm{Id})}$ (see I.2(b) of Theorem 5.1), $s_1 = 1$. We have the following possibilities:

(i) $s_2 = 2$, $M_2 = Mb$ with an action given by

$$(t, y) \to (t + 4i/n, y);$$

then we obtain the action described in II(e) of Theorem 7.1.

(ii) $s_2 = 1$, $M_2 = Mb$ with an action of type $(2;\ 2, A)_{(D^1, A)}$; then we obtain the involution described in II(d).

(iii) $s_2 = 1$, $M = S^1 \times D^1$ with an action given by

$$(t, y) \to \big(t + 1/2 + 2i/n, A(y)\big);$$

then we obtain the involution described in I.2(d).

(b) There exists a 1-sided embedding $S^1 \hookrightarrow \mathrm{Iz}(M) \subset M = B$. We have two possibilities:

(i) $S^1 \neq T(S^1) = S^1_0$. It is easy to see that $B - S^1 - S^1_0$ is connected and equal (after attaching the boundary) to $S^1 \times D^1$. By Theorem 4.1 in [11] and Theorem 5.1, the action of T is as in II(f) of Theorem 7.1.

(ii) $T(S^1) = S^1$. We deduce in the same way as in case (b) (i) that we have to do with the action described in II(g).

We can obtain the classification of actions described in the subcases I.1(a), ..., II(g) similarly as in the previous theorems. Actions described in I.1(c), (d), II(e), (f), and (g) are nonstandard. It follows from Theorem 6.5 and from the fact that each standard action on $S^1 \hat{\times} S^1$ extends to a standard action on $S^1 \hat{\times} D^2$. This completes the proof of Theorems 7.1 and 1.4.

8. Final remarks. We may use the results obtained in this paper to classify actions of Z_n on a connected (and disk, see [7]) sum of manifolds considered in our paper. This problem will be studied in [13].

One can extend Theorem 3.1 and Lemma 3.2 to actions which satisfy condition (*) (Definition 6.1), similarly as in Theorem 6.5. Then, as a particular case, we obtain the classification of involutions on $S^1 \hat{\times} S^2$ (see [4] and [17]).

It is of great interest to verify when condition (*) is true. There are only partial results, e.g., those of Tollefson [18]. (**P 1271**)

Recently, Professor J. Birman has informed me that the Smith Conjecture has been established. "The proof of the Smith Conjecture represents a culmination of the efforts of many mathematicians... The broad outlines of this proof were first brought into focus by W. Thurston."

W. Meeks III and S. T. Yau in *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory* have proved that condition (*) is satisfied for $S^1 \hat{\times} D^2$, $S^1 \hat{\times} P^2$, and $S^1 \hat{\times} S^2$.

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