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LARGE SMALL SETS

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Z. Buczolich asked if there exists a measure-zero, first category set containing a translated copy of every countable set of reals. It is easy to see that every set which is either of full measure or of full category has this latter property, therefore there are such sets with either measure zero or of first category.

A rather easy solution to this problem is the following.

Partition ω into the disjoint union of infinitely many infinite sets as $\omega = \bigcup H_n$. We define a set of reals A as follows. A real x is in A if and only if, for some n, all the 3-ary digits of x in H_n are different from 2. Obviously, A is both measure zero and of first category. In order to show the other property, let $r_0, r_1, \ldots$ be real numbers. Choose a real number r in such a way that if $i \in H_n$ and the i-th digit of r_n is 0, 1 or 2, then the i-th digit of r is 0, 2, 1, respectively. Then the i-th digit of $r+r_n$ is 0 or 1 (if a one is borrowed) for all $i \in H_n$, i.e., $r+r_n \in A$. This construction can be found in [2].

In this paper* we extend this statement under Martin's Axiom. We show that if $MA_\varkappa$ holds (see [1]), then a similar set exists containing a translate for every set of size $\varkappa$.

First we need a generalization of the fact that the union of functions defined on disjoint sets is a function again.

DEFINITION. If S is a set, and μ a cardinal, then $[S]^\mu$ ($[S]^{<\mu}$) denotes the collection of μ-sized ($<\mu$-sized) subsets of S. Two sets A, B are *almost disjoint* if $A \cap B$ is finite.

THEOREM. ($MA_\varkappa$) *If* H_α $(\alpha < \varkappa)$ *are almost disjoint countable sets,* $f_\alpha\colon H_\alpha \to \omega$, *then there is a function*

$$G\colon \bigcup \{H_\alpha\colon \alpha < \varkappa\} \to [\omega]^2$$

such that for every $\alpha < \varkappa$ *the set*

$$\{x \in H_\alpha\colon f_\alpha(x) \notin G(x)\}$$

is finite.

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Notice that by [1] we cannot insure a mapping to $[\omega]^1$.

Proof. We are going to define a partial order. The pair $\langle s, g\rangle$ is in a set P iff $s \in [\varkappa]^{<\omega}$, g is a function from a finite part of $\bigcup\{H_\alpha: \alpha < \varkappa\}$ into $[\omega]^2$, if $\alpha \neq \beta$ are in s, then $H_\alpha \cap H_\beta$ is covered by $\mathrm{Dom}(g)$. Order P as follows: $\langle s', g'\rangle \leqslant \langle s, g\rangle$ iff $s' \supseteq s$, $g' \supseteq g$, and for $x \in \mathrm{Dom}(g') - \mathrm{Dom}(g)$, $\alpha \in s$, if $x \in H_\alpha$, then $f_\alpha(x) \in g'(x)$ holds. This is clearly a partial order.

CLAIM 1. *For $\alpha < \varkappa$ the set $\{\langle s, g\rangle: \alpha \in s\}$ is dense.*

Proof. To be able to extend an $\langle s, g\rangle$ to $\langle s \cup \{\alpha\}, g'\rangle$, we need to extend g to the points of the set

$$A = \bigcup\{H_\alpha \cap H_\beta: \beta \in s\} - \mathrm{Dom}(g).$$

For $x \in A$ there is exactly one such $\beta \in s$, so we can choose

$$g'(x) \supseteq \{f_\alpha(x), f_\beta(x)\}.$$

CLAIM 2. *For every $x \in \bigcup\{H_\alpha: \alpha < \varkappa\}$ the set*

$$\{\langle s, g\rangle: x \in \mathrm{Dom}(g)\}$$

is dense.

Proof. We can assume that $\langle s, g\rangle \in P$, $\alpha \in s$ (by Claim 1), $x \in H_\alpha$. If $x \notin \mathrm{Dom}(g)$, then only this α can have $x \in H_\alpha$, so we only have to insure $g'(x) \ni f_\alpha(x)$ and nothing else.

CLAIM 3. *If $\langle s_1, g_1\rangle$, $\langle s_2, g_2\rangle \in P$ are such that*

(i) *$g_1 \cup g_2$ is a function,*

(ii) *for $\alpha \in s_1$ we have $H_\alpha \cap (\mathrm{Dom}(g_2) - \mathrm{Dom}(g_1)) = \emptyset$,*

(iii) *for $\alpha \in s_2$ we have $H_\alpha \cap (\mathrm{Dom}(g_1) - \mathrm{Dom}(g_2)) = \emptyset$,*

then $\langle s_1, g_1\rangle$ and $\langle s_2, g_2\rangle$ are compatible.

Proof. Take $\langle s, g\rangle$, where $s = s_1 \cup s_2$, g' being $g_1 \cup g_2$, g extends g' to those points x for which $x \notin \mathrm{Dom}(g_1 \cup g_2)$, and there exist $\alpha \in s_1$, $\beta \in s_2$, $x \in H_\alpha \cap H_\beta$. These α, β must be unique (as $x \notin \mathrm{Dom}(g_1) \cup \mathrm{Dom}(g_2)$), so we can find $g(x) \supseteq \{f_\alpha(x), f_\beta(x)\}$. As g is a function, $\langle s, g\rangle$ is a condition. To see, e.g., that $\langle s, g\rangle \leqslant \langle s_1, g_1\rangle$, if $x \in \mathrm{Dom}(g) - \mathrm{Dom}(g_1)$, $x \in H_\alpha$, $\alpha \in s_1$, by (ii), $g(x)$ is defined to contain $f_\alpha(x)$.

CLAIM 4. *If $\{u_n: n < \omega\}$ are disjoint k-element sets, $k < \omega$, and $\{A_0, A_1, \ldots, A_k\}$ are almost disjoint countable sets, then there exist i, j with $u_i \cap A_j = \emptyset$.*

Proof. As the A_j's are almost disjoint and the u_i's are disjoint, only finitely many of the u_i's can contain elements from two or more A_j's. So there is a u_i such that the A_j's must meet it in disjoint sets, and as there are $k+1$ A_j's and u_i has only k elements, there is an A_j disjoint from u_i.

CLAIM 5. *P satisfies the ccc.*

Proof. Assume that $\langle s_\nu, g_\nu\rangle \in P$ for $\nu < \omega_1$. Without loss of generality we can assume at once that $\{s_\nu: \nu < \omega_1\}$ and $\{\mathrm{Dom}(g_\nu): \nu < \omega_1\}$ both form Δ-systems, $s_\nu = t \cup r_\nu$, $\mathrm{Dom}(g_\nu) = w \cup v_\nu$, $|v_\nu| = k$, $g_\nu | w$ are the same. By shrinking, if necessary, for $\nu < \mu < \omega_1$ and for every $\alpha \in r_\nu$ we can achieve

$$(*) \qquad (H_\alpha - w) \cap v_\nu = \emptyset.$$

SUBCLAIM. *There are $\nu < \mu < \omega_1$ such that if $\alpha \in r_\mu$, then $(*)$ holds.*

Proof. Choose $u_i = v_i$ for $i < \omega$, and $A_j = \bigcup \{H_\alpha: \alpha \in r_{\omega+j}\}$; then apply Claim 4.

To complete the proof of Claim 5 one has only to observe that $\langle s_\nu, g_\nu\rangle$, $\langle s_\mu, g_\mu\rangle$ satisfy the conditions of Claim 3.

The proof of the Theorem follows by the usual methods using our claims (see [1]).

COROLLARY. ($\mathrm{MA}_\varkappa$) *There exists a measure-zero, first category set containing a translated copy for every set of size $\varkappa$.*

Proof. Let $H_\alpha \in [\omega]^\omega$ be almost disjoint sets for $\alpha < \varkappa$. Let $x \in A_\alpha$ if, for $i \in H_\alpha$ large enough, the i-th 5-ary digit of x is different from 4. By $\mathrm{MA}_\varkappa$, the set $A = \bigcup \{A_\alpha: \alpha < \varkappa\}$ is both measure zero and of first category. Assume that x_α $(\alpha < \varkappa)$ are real and let $x_\alpha(i)$ be the i-th 5-ary digit of x_α. By the Theorem, there is a $g: \omega \to [5]^2$ such that, for every α, if $n \in H_\alpha$ is large enough, then $x_\alpha(n) \in g(n)$. Let us choose $h: \omega \to 5$ in such a way that $g(n) + h(n) \subseteq \{0, 1, 2\}$ with addition modulo 5. Consequently, the n-th digit of $x_\alpha + h$ is in $\{0, 1, 2, 3\}$ (one 1 may be borrowed), i.e., $x_\alpha + h \in A$.

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