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CORRECTION TO THE PAPER "A THEOREM ON ALMOST DISJOINT SETS"

(Colloquium Mathematicum 24 (1971), p. 1-2)

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The following is a correct version of the result in that paper:

THEOREM. *Let M be an infinite set of power a, and $f_i\colon M \to M$ an injection for each $i < a$. If the set $F_i = \{x \in M\colon f_i(x) = x\}$ of fixed points of f_i has the power less than a and the set $F_{ij} = \{x \in M\colon f_i(x) = f_j(x)\}$ of points at which f_i and f_j agree has the power less than a, then there exists a subset $X \subseteq M$ of power a such that any two members of the sequence*

$$X, f_0(X), f_1(X), \ldots, f_i(X), \ldots \quad (i < a)$$

are almost disjoint.

Proof. We define the required set $X = \{x_k\colon k < a\}$ by induction on k. Specifically, let x_0 be an arbitrary member of M and suppose that x_i for $i < k$ have all been defined. Let x_k be any member of M which satisfies

$$x_k \notin X_k \cup \bigcup_{s<k} f_s^{-1}(X_k) \cup \bigcup_{s<k} f_s(X_k) \cup \bigcup_{s,t<k} f_s^{-1} f_t(X_k),$$

where $X_k = \{x_i\colon i < k\}$.

We show first that

$$f_s(x_k) \neq x_j \quad \text{for } j \neq k \text{ if } s \leqslant k. \tag{1}$$

Suppose $f_s(x_k) = x_j$ for $j > k$. Then, since $s < j$, the condition

$$x_j \notin \bigcup_{s \leqslant j} f_s(X_j)$$

applies, so we have $x_k \notin X_j$, and so $k \geqslant j$, contrary to our supposition.

Now suppose $f_s(x_k) = x_j$ for $j < k$, i.e., $x_k = f_s^{-1}(x_j)$ and, therefore, since $s \geqslant k$, the condition

$$x_k \notin \bigcup_{s \leqslant k} f_s^{-1}(X_k)$$

applies, so that we have $x_j \notin X_k$, i.e., $j \geqslant k$, which is again contrary to our supposition. Thus (1) holds.

We now show that

$$(2) \qquad |f_s(X) \cap X| < a \quad \text{for all } s < a.$$

Suppose otherwise that, for some $s < a$, we have $f_s(x_{k(i)}) = x_{j(i)}$ for all $i < a$. By (1), this means that $k(i) = j(i)$ for all $k(i), j(i) \geqslant s$, which contradicts $|F_s| < a$, and so (2) holds.

Finally, we show that

$$(3) \qquad |f_s(X) \cap f_t(X)| < a \quad \text{for all } s, t < a,\ s \neq t.$$

Again we argue indirectly and suppose that, for some distinct $s, t < a$, (3) does not hold. Thus we suppose that

$$(4) \qquad f_s(x_{k(i)}) = f_t(x_{j(i)}) \quad \text{for all } i < a.$$

Put $i_0 = \max(x, t)$ and assume that $k(i), j(i) > i_0$. Then the conditions $x_{j(i)} \notin f_t^{-1} f_s(X_{j(i)})$ and $x_{k(i)} \notin f_s^{-1} f_t(X_{k(i)})$ hold. We claim that $x_{k(i)} \notin X_{j(i)}$. Otherwise we would have, in view of (4), $f_t(x_{j(i)}) \in f_s(X_{j(i)})$, which contradicts the first of these conditions. Similarly, the second one yields $x_{j(i)} \notin X_{k(i)}$. Consequently, $k(i) \geqslant j(i)$ and $j(i) \geqslant k(i)$. Thus (4) cannot hold if $k(i), j(i) > i_0$ unless $k(i) = j(i)$ for all $i > i_0$ which contradicts the condition that $|F_{st}| < a$. On the other hand, if only $k(i) > i_0$ and $j(i) \leqslant i_0$, then (4) implies that f_s is not an injection and, similarly, if $j(i) > i_0$ and $k(i) \leqslant i_0$, then (4) implies that f_t is not an injection. Thus (4) can only hold if $k(i), j(i) \leqslant i_0$ which we assumed not to be the case, and so the Theorem is proved.

In the paper quoted in the title an erroneous proof of the stronger result, which assumes only that $|M - F_{ij}| = a$ for all $i, j < a$ rather than $|F_{ij}| < a$, was given. I do not know whether this stronger result is true. (**P 937**)

The converse of this stronger result is false, contrary to the assertion made there.

I am very grateful to D. H. Pelletier for bringing both these points to my attention.

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