COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 2

THE HOMOTOPY CATEGORY OF CHAIN COMPLEXES IS A HOMOTOPY CATEGORY

BY

MAREK GOLASIŃSKI AND GRZEGORZ GROMADZKI (TORUŃ)

In this paper we show that the category of chain complexes (not necessarily bounded) over an abelian category with fibrations and cofibrations in the sense of Kamps (see [2]) and with the chain homotopy equivalences as weak equivalences is the closed model category in the sense of Quillen (see [4]).

Moreover, we prove that this category is a proper closed model category (see [1]).

1. Preliminaries. Let $d\mathscr{A}$ be the category of chain complexes over an abelian category $\mathscr{A}$. In [3] Kamps proved the following

PROPOSITION. *A chain map $f\colon X \to Y$ is a fibration iff it is a normal epimorphism (i.e. $f_n\colon X_n \to Y_n$ is a retraction in $\mathscr{A}$ for each integer n).*

He announced that the dual is also true.

In his proof Kamps used a *homotopy system* (I, j_0, j_1, q) in $d\mathscr{A}$ (see [2]) defined as follows:

let X be a chain complex and $f\colon X \to Y$ a chain map,

$$IX_n = X_n \oplus X_n \oplus X_{n-1} \quad \text{and} \quad If_n = f_n \oplus f_n \oplus f_{n-1}.$$

The boundary d^{IX} of IX is given by the matrix

$$d_n^{IX} = \begin{pmatrix} d_n^X & 0 & \mathrm{id}_{X_{n-1}} \\ 0 & d_n^X & -\mathrm{id}_{X_{n-1}} \\ 0 & 0 & -d_{n-1}^X \end{pmatrix},$$

and j_0^X, j_1^X, q^X are determined by the matrices

$$\begin{pmatrix} \mathrm{id}_{X_n} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathrm{id}_{X_n} \\ 0 \end{pmatrix}, \quad (\mathrm{id}_{X_n}\ \mathrm{id}_{X_n}\ 0),$$

respectively.

Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j_0^X}\downarrow & & \downarrow{\scriptstyle k} \\ IX & \xrightarrow{l} & Z_f \end{array}$$

be a pushout in $d\mathscr{A}$ (Z_f is called the *mapping cylinder* of f). We have

$$(Z_f)_n = Y_n \oplus X_n \oplus X_{n-1} \quad \text{and} \quad d_n^{Z_f} = \begin{pmatrix} d_n^Y & 0 & f_{n-1} \\ 0 & d_n^X & -\mathrm{id}_{X_{n-1}} \\ 0 & 0 & -d_{n-1}^X \end{pmatrix}.$$

The maps k and l are defined by

$$k_n = \begin{pmatrix} \mathrm{id}_{Y_n} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad l_n = f_n \oplus \mathrm{id}_{X_n} \oplus \mathrm{id}_{X_{n-1}}.$$

A *cohomotopy system* (P, p_0, p_1, s) in $d\mathscr{A}$ (see [2]) is defined similarly. It is not difficult to see that chain homotopies

$$(s_n\colon X_n \to Y_{n+1})_{n\in Z}$$

of the chain maps $f, g\colon X \to Y$ are in 1-1 correspondence with chain maps $H\colon IX \to Y$ such that $H\cdot j_0^X = f$ and $H\cdot j_1^X = g$ (or $H\colon X \to PY$ such that $p_0^Y\cdot H = f$ and $p_1^Y\cdot H = g$). Write $H\colon f \simeq g$.

The chain maps $f, g\colon X \to Y$ are called *homotopic over* $i\colon Y \to Z$ (respectively, *under* $p\colon W \to X$) if $i\cdot f = i\cdot g$ (respectively, $f\cdot p = g\cdot p$) and there exists a homotopy $H\colon IX \to Y$ from f to g such that the diagrams

$$\begin{array}{ccc} IX & \xrightarrow{H} & Y \\ {\scriptstyle I(i\cdot f)}\downarrow & & \downarrow{\scriptstyle i} \\ IZ & \xrightarrow{q^Z} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} IW & \xrightarrow{q^W} & W \\ {\scriptstyle Ip}\downarrow & & \downarrow{\scriptstyle f\cdot p} \\ IX & \xrightarrow{H} & Y \end{array}$$

respectively, are commutative. We write $H\colon f \overset{i}{\simeq} g$ (respectively, $H\colon f \underset{p}{\simeq} g$).

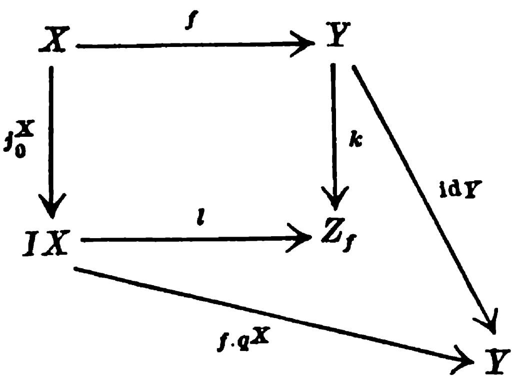
Remark. It is not difficult to check that the chain maps $f, g\colon X \to Y$ are homotopic over $i\colon Y \to Z$ (respectively, under $p\colon W \to X$) iff $i\cdot f$

$= i \cdot g$ (respectively, $f \cdot p = g \cdot p$) and there exists a chain homotopy $(s_n\colon X_n \to Y_{n+1})_{n \in Z}$ from f to g such that $i_{n+1} \cdot s_n = 0$ (respectively, $s_n \cdot p_n = 0$) for each integer n.

2. The main theorem. A chain map $f\colon X \to Y$ is called a *fibration* (*cofibration*) if it is a fibration (cofibration) in the sense of Kamps (see [2]) and f is said to be a *weak equivalence* if it is a chain homotopy equivalence.

LEMMA 1. *Any chain map $f\colon X \to Y$ may be factored $f = p \cdot i$, where i is a cofibration and p is a trivial fibration.*

Proof. Consider the commutative diagram



in the category of chain complexes.

There exists a chain map $p\colon Z_f \to Y$ such that $p \cdot l = f \cdot q^X$ and $p \cdot k = \mathrm{id}_Y$.

For $i = l \cdot j_1^X$ we have $p \cdot i = f$.

Since

$$p \cdot k = \mathrm{id}_Y \quad \text{and} \quad i = \begin{pmatrix} 0 \\ \mathrm{id}_X \\ 0 \end{pmatrix},$$

p is a fibration and i is a cofibration by the result of Kamps (see [3]).

In order to prove that p is a weak equivalence it suffices to show that $k \cdot p \simeq \mathrm{id}_{Z_f}$ since $p \cdot k = \mathrm{id}_Y$.

It is easy to check that the sequence of maps

$$s_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{id}_{X_n} & 0 \end{pmatrix}\colon (Z_f)_n \to (Z_f)_{n+1}$$

is the required homotopy.

Note that the dual lemma is also true.

LEMMA 2. *The following statements are equivalent for a chain map $f\colon X \to Y$:*

(i) *f is a trivial fibration;*

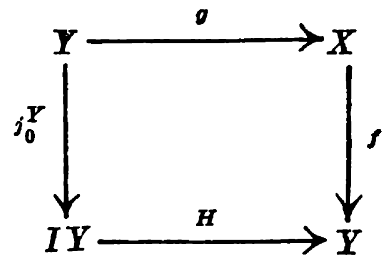
(ii) *f has the RLP* (see [4]) *with respect to the cofibrations*;

(iii) *f is a fibration and a strong deformation coretract* (*i.e. there exists a chain map r: $Y \to X$ such that $f \cdot r = \mathrm{id}_Y$ and $r \cdot f \underset{f}{\simeq} \mathrm{id}_X$*).

Proof. (iii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Let $g: Y \to X$ be a chain map such that $f \cdot g \simeq \mathrm{id}_Y$, $g \cdot f \simeq \mathrm{id}_X$ and let H: $f \cdot g \simeq \mathrm{id}_Y$.

Consider the commutative diagram



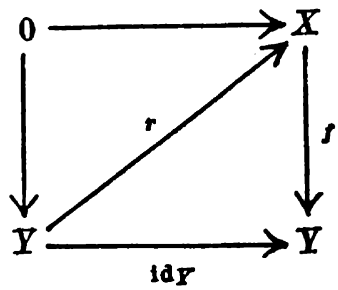
Since f is a fibration, there exists a chain map G: $IY \to X$ such that $f \cdot G = H$ and $G \cdot j_0^Y = g$.

For $r = G \cdot j_1^Y$ we have $f \cdot r = \mathrm{id}_Y$ and $g \simeq r$, so $r \cdot f \simeq \mathrm{id}_X$. Let F: $r \cdot f \simeq \mathrm{id}_X$. Then F': $IX \to X$ given by the sequence of maps

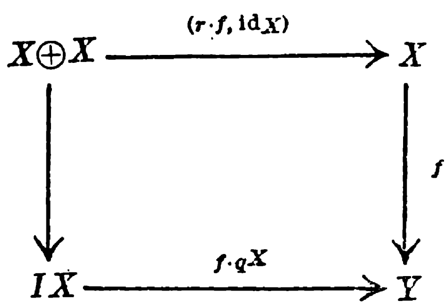
$$F'_n = F_n - r_n \cdot f_n \cdot F_n \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{X_{n-1}} \end{pmatrix}$$

is the required homotopy.

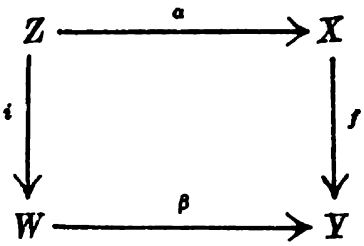
(ii) $\Rightarrow$ (iii). The chain map $0 \to Y$ is a cofibration, so there exists a chain map r: $Y \to X$ such that the diagram



commutes. The chain map $X \oplus X \to IX$ induced by j_0^X and j_1^X is a cofibration by the result of Kamps. Therefore, the strong deformation may be constructed by lifting in the diagram



(iii) ⇒ (ii). A lifting in the diagram



where i is a cofibration, can be constructed in the following way.

Let $Z' = \operatorname{coker} i$. Without loss of generality we can assume that

$$W_n = Z_n \oplus Z'_n, \qquad \beta_n = (\beta_n^1, \beta_n^2), \qquad \text{and} \qquad i = \begin{pmatrix} \mathrm{id}_Z \\ 0 \end{pmatrix}.$$

Then the boundary d_n^W of W takes the form

$$d_n^W = \begin{pmatrix} d_n^Z & \theta_n \\ 0 & d_n^Z \end{pmatrix},$$

where $\theta_n\colon Z'_n \to Z_{n-1}$ is a suitable morphism of $\mathscr{A}$.

By assumption there is a chain map $r\colon Y \to X$ such that $f\cdot r = \mathrm{id}_Y$ and $H\colon r\cdot f \overset{f}{\simeq} \mathrm{id}_X$. Let $s_n\colon X_n \to X_{n+1}$ be a sequence of maps induced by H. Then the sequence of maps $t_n\colon Z_n \oplus Z'_n \to X_n$ given by the matrices

$$t_n = (\alpha_n \quad r_n\cdot\beta_n^1 - s_{n-1}\cdot\alpha_{n-1}\cdot\theta_n)$$

induces a lifting in the above diagram.

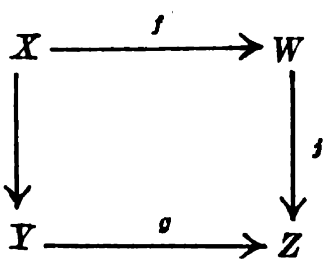
A suitable dualization of Lemma 2 is also true.

THEOREM (the main result). *The category $d\mathscr{A}$ defined as above is a closed model category.*

Proof. It suffices to verify axioms M0, M2, M5 and M6 (see [4]). M0 is clear. M2 follows directly from Lemma 1 and from its dualization. M5 is obvious. M6 follows immediately from Lemmas 1 and 2.

From the Theorem and the following lemma we infer that $d\mathscr{A}$ is also a proper closed model category.

LEMMA 3. *If the square*



is a pushout in $d\mathscr{A}$ with i a cofibration and f a weak equivalence, then g is a weak equivalence, and if the square is a pullback with j a fibration and g a weak equivalence, then f is a weak equivalence.

Proof. Without loss of generality we can assume that

$$Y_n = X_n \oplus X'_n, \quad i_n = \begin{pmatrix} \mathrm{id}_{X_n} \\ 0 \end{pmatrix}, \quad \text{and} \quad d_n^Y = \begin{pmatrix} d_n^X & \theta_n \\ 0 & d_n^X \end{pmatrix},$$

where $X = \operatorname{coker} i$.

It is easy to check that $Z_n = W_n \oplus X'_n$, and

$$j_n = \begin{pmatrix} \mathrm{id}_{W_n} \\ 0 \end{pmatrix}, \quad g_n = \begin{pmatrix} f_n & 0 \\ 0 & \mathrm{id}_{X_n} \end{pmatrix}, \quad d_n^Z = \begin{pmatrix} d_n^W & f_{n-1}\cdot\theta_n \\ 0 & d_n^X \end{pmatrix}$$

for each integer n.

Let the chain map $h\colon W \to X$ be the homotopy inverse for f and let $H\colon h\cdot f \simeq \mathrm{id}_X$, $G\colon f\cdot h \simeq \mathrm{id}_W$. Then the chain map $k\colon Z \to Y$ given by the sequence

$$k_n = \begin{pmatrix} h_n & 0 \\ 0 & \mathrm{id}_{X_n} \end{pmatrix}$$

is the homotopy inverse for g.

The suitable homotopies are established by H and G in an obvious way. This completes the proof.

The second part can be proved similarly and we omit it.

REFERENCES

[1] A. K. Bousfield and E. M. Friendlander, *Homotopy theory of Γ-spaces, spectra and bisimplicial sets*, p. 80-130 in: *Geometric applications of homotopy theory, III* (Proceedings of the Conference, Evanston, Illinois, 1977), Lecture Notes in Mathematics 658 (1978).

[2] K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopietheorie*, Mathematische Zeitschrift 124 (1972), p. 215-236.

[3] — *Note on normal sequences of chain complexes*, Colloquium Mathematicum 39 (1978), p. 225-227.

[4] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics 43 (1967)

INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY
TORUŃ

Reçu par la Rédaction le 14. 11. 1979