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ON 2-RAMIFICATION POINTS OF A DENDROID

BY

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In [2] Charatonik published results of a study of ramification points of continua and pointed out some applications of those results to dendroids. Lelek [4] has studied endpoints of continua and applied his results to dendroids. Charatonik has also defined the degree of non-local connectedness of continua and has published [3] results concerning the manipulative and computational properties of the degree of non-local connectedness. The author is, however, unaware of any work which gives a significant amount of consideration to the points of a dendroid at which the dendroid is not locally connected. In this paper*, the author defines a 2-ramification point of a dendroid which gives a useful approach to the study of the points at which a dendroid is not locally connected. Results are presented to establish a sufficient condition that a dendroid contain a 2-ramification point, and a subset of any dendroid will be identified which is the smallest subset of the dendroid which contains the 2-ramification points and which is necessarily closed.

In this paper, all spaces are *metric*, a *continuum* is a compact, connected space, and an *arc* is a non-degenerate continuum with no more than two non-separating points. A *dendroid* is a hereditarily unicoherent, arcwise connected continuum. The point p is a *ramification point* of a dendroid if the dendroid contains three arcs such that the intersection of each two of them contains only p. The point p is a *terminal point* of a dendroid M if p is an endpoint of every arc containing p which is contained in M. In a dendroid there is only one arc between each two points; hence, if p and q are points, then $[p, q]$ will denote the arc from p to q. If M is a set, $\bar{M}$ denotes the closure of M. A space M is said to be *connected im kleinen* at the point p of M if it is true that for each open set U containing p there is an open set V containing p such that V is a subset of U and such that if q is a point of V, then some connected subset of U contains both p and q. If p is a point and ε is a positive number, $S(p, \varepsilon)$ denotes the set of all points x such that the distance from p to x is less than ε.

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1. The existence of a 2-ramification point. The point p is defined to be a *2-ramification point* of a dendroid M if p is not a terminal point of M and if there exists a sequence $p_1, p_2, p_3, \ldots$ of points of M which converges to p such that no two points of the sequence are contained in the same subarc of M with p.

Example 1.1. Let M denote the union of $[-1, 1]\times 0, 0\times[0, 1]$ and $t\times[0, 1]$ for $t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ M is a dendroid and each point of $0\times[0, 1)$ is a 2-ramification point of M. The point $(0, 0)$ is a cutpoint of M, but is the only 2-ramification point which is a cutpoint. Also, $(0, 0)$ is the only 2-ramification point which is a ramification point of M.

THEOREM 1.1. *If a dendroid M is not locally connected, then M contains a 2-ramification point.*

Proof. Since M is not locally connected, there is a point p of M at which M is not connected im kleinen. There is a positive number $\varepsilon_1 \leqslant 1$ such that if $D_1 = S(p, \varepsilon_1)$, then D_1 is an open set containing p with the property that if D' is any open set containing p and contained in D_1, then D' is not connected. The set D_1 is not connected and has infinitely many components. Let C denote the component of D_1 which contains p. Let C_1 denote a component of D_1 distinct from C, let p_1 denote a point of C_1 and let B_1 denote the arc $[p, p_1]$. Consequently, sequences $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, p_1, p_2, p_3, \ldots,$ $C_1, C_2, C_3, \ldots, D_1, D_2, D_3, \ldots$ and $B_1, B_2, B_3, \ldots$ can be constructed with the following properties:

(1) $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is a sequence of positive numbers which converges to 0;

(2) for $i > 1$, D_i is an open set containing p, D_i is a subset of $S(p, \varepsilon_i)$, $\overline{S(p, \varepsilon_i)}$ is a subset of D_{i-1} and D_i does not intersect C_{i-1};

(3) for $i > 1, C_i$ is a component of $D_1-(C\cup C_1\cup C_2\cup \ldots \cup C_{i-1})$;

(4) for $i > 1, p_i$ is a point of $C_i\cap D_i$;

(5) for $i > 1, B_i$ is the arc $[p, p_i]$ in M and

(6) for $i > 1, D_i\cap B_{i-1}$ is connected.

If i and j are positive integers, $i > j$, D_i is a subset of D_{j+1} which contains p_i. Since $D_i\cap B_j$ is a subset of $D_{j+1}\cap B_j$, a connected subset of D_1 which contains p, and since p_i and p are is distinct components of D_1, p_i is not in B_j.

There does not exist an infinite subsequence $B_{n_1}, B_{n_2}, B_{n_3}, \ldots$ of $B_1, B_2, B_3, \ldots$ such that B_{n_i} is a subset of $B_{n_{i+1}}$ for each i. If B_{n_i} is a subset of $B_{n_{i+1}}$ for each i, then the closure of $B_{n_1}\cup B_{n_2}\cup B_{n_3}\cup \ldots$ is an arc ([1], p. 18]). Now, p is the sequential limit point of $p_{n_1}, p_{n_2}, p_{n_3}, \ldots$; hence, if U is any open set containing p such that U is a subset of D_1, there is a positive integer m such that, if $j > m$, p_{n_j} is contained in U. Since $p_{n_m}, p_{n_{m+1}}, p_{n_{m+2}}, \ldots$ are in distinct components of D_1; they are in distinct components of U. Hence, $\overline{B_{n_1}\cup B_{n_2}\cup B_{n_3}\cup \ldots}$ is not locally connected, a contradiction to the well known fact that an arc is locally connected.

Based on the two preceding paragraphs, a subsequence $A_1, A_2, A_3, \ldots$ of $B_1, B_2, B_3, \ldots$ can be selected such that if $i \neq j$, then A_i is not a subset of A_j and A_j is not a subset of A_i. For simplicity, p_i will now denote the endpoint of A_i distinct from p for each positive integer i.

One possibility is that there is an infinite subsequence $A_{n_1}, A_{n_2}, A_{n_3}, \ldots$ of $A_1, A_2, A_3, \ldots$ such that each two elements in the sequence intersect only at the point p. Then, for each i, let r_i denote the last point in A_{n_i} in the cutpoint order from p to p_{n_i} which is in the boundary of D_1 with respect to A_{n_i}. Let J_i denote the arc $[r_i, p_{n_i}]$. Some subsequence J_{m_1}, $J_{m_2}, J_{m_3}, \ldots$ of $J_1, J_2, J_3, \ldots$ can be chosen so that $r_{m_1}, r_{m_2}, r_{m_3}, \ldots$ has a sequential limit point and such that $J_{m_1}, J_{m_2}, J_{m_3}, \ldots$ has a non-degenerate sequential limiting set L which is a subcontinuum of M containing p and the sequential limit point r of the sequence $r_{m_1}, r_{m_2}, r_{m_3}, \ldots$ Since p is in D_1, $p \neq r$. Let t denote any cutpoint of the arc $[p, r]$ in L. Then there is a sequence $t_1, t_2, t_3, \ldots$ of points such that $t_i \in J_{m_i}$, for each i, and such that t is the sequential limit point of $t_1, t_2, t_3, \ldots$ Each arc $[t, t_i]$ contains p as a cutpoint except for possibly one, so infinitely many of the points in the sequence $t_1, t_2, t_3, \ldots$ are such that no two are in the same arc with t. Thus t is a 2-ramification point of M.

Suppose that there is no infinite subsequence of $A_1, A_2, A_3, \ldots$ each two elements of which intersect only at p. Then there is an infinite subsequence $A_{m_1}, A_{m_2}, A_{m_3}, \ldots$ of $A_1, A_2, A_3, \ldots$ such that the intersection of each two elements is non-degenerate. For each positive integer i, let r_i denote the last point of A_{m_i} in the cutpoint order from p to p_{m_i} which is in the boundary of D_1 with respect to A_{m_i}, and let H_i denote the arc $[r_i, p_{m_i}]$. Some infinite subsequence $H_{j_1}, H_{j_2}, H_{j_3}, \ldots$ of $H_1, H_2, H_3, \ldots$ has a sequential limiting set L' containing p and some point r, $r \neq p$, which is the sequential limit point of the sequence $r_{j_1}, r_{j_2}, r_{j_3}, \ldots$ If there is a point s in $[p, r] - p$ such that the intersection of each two of infinitely many of the elements of the sequence $A_{m_{j_1}}, A_{m_{j_2}}, A_{m_{j_3}}, \ldots$ contains the arc $[p, s]$, let t' denote any cutpoint of $[p, s]$. If no such point s exists, let t' denote any cutpoint of $[p, r]$ which is in C. In either case, t' is a 2-ramification point of M.

Example 1.2. Suppose M is the subset of the plane consisting of the union of $[-1, 1] \times 0$ and each of $t \times [0, t]$ for $t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Then M is a locally connected dendroid, the point $(0, 0)$ is a 2-ramification point of M and, clearly, the converse of Theorem 1.1 does not hold.

The following theorem is a consequence of the argument to Theorem 1.1:

THEOREM 1.2. *If a dendroid M is not connected im kleinen at a point p, then p is either a 2-ramification point of M or p is an endpoint of an arc A such that every point of $A-p$ is a 2-ramification point of M.*

THEOREM 1.3. *If a dendroid M is not locally connected at a point p, then p is either a 2-ramification point of M or p is an endpoint of an arc A such that each point of $A-p$ is a 2-ramification point of M.*

Proof. If M is not connected im kleinen at p, Theorem 1.2 gives the conclusion. If there is an open set U containing p such that p is a limit point of $U-C$, where C is the component of U which contains p then an argument can be made in a manner similar to the proof of Theorem 1.1.

Assume that M is connected im kleinen at p. Then, if U is an open set containing p and C is the component of U which contains p, p is not a limit point of $U-C$. Suppose, further, that p is neither a 2-ramification point of M nor an endpoint of an arc such that each point of the arc other than p is a 2-ramification point of M. Also, suppose that p is not a terminal point of M, for, if p is a terminal point of M, then a dendroid M' could be constructed containing an arc B such that $M' = M \cup B$, $M \cap B$ contains only p and any 2-ramification point of M' would be a 2-ramification point of M.

Let Q denote the set of points at each of which M is not connected im kleinen. Since M is not locally connected at p, any open set which contains p contains an open set V containing p such that, if V' is an open set containing p and V' is a subset of V, then V' is not connected. Let C_V denote the component of V which contains p. Now, M is connected im kleinen at p, so p is not a limit point of $V-C_V$. However, some point q of C_V must be a limit point of $V-C_V$ else C_V is a connected open set containing p and contained in V. Then any open set containing q and contained in V must intersect infinitely many components of V. Thus M is not connected im kleinen at q, and p is a limit point of Q. Since p is not an endpoint of an arc consisting of 2-ramification points of M, no sequence of points of Q converges to p if infinitely many of them are all on a single arc with p, else a construction can easily be made that will show p to be a 2-ramification point of M. For the same reason, there is no sequence of ramification points of M which will converge to p if infinitely many are all on a single arc with p.

For each terminal point t in M, let x_t denote the point distinct from p in $[t, p]$ which is the last point of $[t, p]$ in the cutpoint order from t to p which is either a ramification point of M or a point of Q. Of course, if $[t, p]$ contains no ramification point of M different from p, let $x_t = t$. Let H denote the set to which h belongs if and only if h is x_t for some terminal point t of M. Then, for each $h \in H$, let M_h denote the set to which x belongs if and only if x is in $[h, p]$ or h is in $[x, p]$. Notice that if h and j are in H and $h \neq j$, then $M_h \cap M_j$ contains only p. Since p is a limit point of Q, there are infinitely many points of H. Also, for each $h \in H$, M_h is a dendroid which is locally connected at p.

Under the assumption that p is not an endpoint of an arc consisting of 2-ramification points of M, the boundary of any open set containing p intersects at most a finite number of the sets M_h for h in H. Let U denote an open set containing p and let $M_{h_1}, M_{h_2}, \ldots, M_{h_n}$ denote all of the sets M_h which intersect $\bar{U} - U$. Then, if $h \neq h_i$ for $i = 1, 2, \ldots, n$, M_h is contained in U. There is an open set V containing p such that V is a subset of U and $V \cap (M_{h_1} \cup M_{h_2} \cup \ldots \cup M_{h_n})$ is connected. For each point q in $M - (M_{h_1} \cup M_{h_2} \cup \ldots \cup M_{h_n})$ there is an open set V_q containing q such that V_q does not intersect $M_{h_1} \cup M_{h_2} \cup \ldots \cup M_{h_n}$. If V' is the set to which y belongs if and only if y is a point of V or y is a point of V_q for some point q as above, then V' is a connected open set containing p and contained in U. Thus, M is locally connected at p which contradicts the hypothesis to the theorem. Hence, p is either a 2-ramification point of M or an endpoint of an arc A such that each point of $A - p$ is a 2-ramification point of M.

THEOREM 1.4. *If a dendroid M is not locally connected at a point p, then p is either a ramification point, a 2-ramification point or a terminal point of M.*

This follows from the three preceding theorems.

2. A closed set containing 2-ramification points. In this section it will be established that the set consisting of the ramification points, 2-ramification points and terminal points of dendroid is closed.

LEMMA 2.1. *If p is a limit point of the set of 2-ramification points of a dendroid M and p is not a 2-ramification point of M, then p is either a terminal point or a ramification point of M.*

Proof. Suppose that p is neither a terminal point nor a ramification point of M. Then p is a cutpoint of an arc $[x, y]$; and, if A is an arc containing p, then the intersection of A and $[x, y]$ is non-degenerate. Also, by Theorem 1.4, M is locally connected at p. So, since p is a limit point of the set of 2-ramification points, sequences $p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots$ and $r_1, r_2, r_3, \ldots$ of points of M can be constructed each of which converges to p such that, for each i, p_i is a 2-ramification point of M, q_i is a point (given by the definition of a 2-ramification point) in a sequence converging to p_i which is not in $[x, y]$ or in $[p, q_j]$ for $j < i$, r_i is the last point of $[p, q_i]$ in the cutpoint order from p to q_i which is also in $[x, y]$, and $r_i \neq r_j$ if $i \neq j$. Each point r_i is a point of $[x, p]$ or of $[p, y]$, and $r_i \neq p$ for each i. Let L denote the set to which r belongs if and only if r is r_i for some i and r_i is in $[x, p]$, and let R denote the set to which r belongs if and only if r is r_i for some i and r_i is in $[p, y]$. One of L or R is infinite. Suppose, then, that L contains each of $r_{n_1}, r_{n_2}, r_{n_3}, \ldots$, a subsequence of $r_1, r_2, r_3, \ldots$ If i and j are integers, $B = [q_{n_i}, r_{n_i}] \cup [r_{n_i}, r_{n_j}] \cup [r_{n_j}, q_{n_j}]$ is an arc containing q_{n_i} and q_{n_j} but not p, and each of $B \cap [p, q_{n_i}]$ and $B \cap [p, q_{n_j}]$ is non-degenerate. Then $q_{n_1}, q_{n_2}, q_{n_3}, \ldots$ is a sequence of points of M converging to p, no two of which are in the same arc with p. Then p is a 2-ramification point of M. Since this is a contradiction to the hypothesis, p is either a ramification point or a terminal point of M.

LEMMA 2.2. *If p is a limit point of the set of ramification points of a dendroid M and p is not a ramification point of M, then p is either a terminal point or a 2-ramification point of M.*

Proof. Suppose p is not a terminal point of M. If M is not locally connected at p, then, by Theorem 1.4, p is a 2-ramification point and the lemma is proved. Suppose, then, that M is locally connected at p. Let x and y denote distinct points of M such that p is a cutpoint of $[x, y]$. Since p is not a ramification point of M, $A \cap [x, y]$ is non-degenerate for each arc A containing p. Then sequences $p_1, p_2, p_3, \cdots, q_1, q_2, q_3, \cdots$ and $r_1, r_2, r_3, \ldots$ of points of M can be constructed so that each of $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ converges to p and such that, for each i, p_i is a ramification point of M, q_i is a point not in $[x, y]$, and r_i is the last point of $[p, q_i]$ in the cutpoint order from p to q_i which is contained in $[x, y]$. Since M is locally connected at p, these sequences can also be constructed so that, for each i, r_i is a cutpoint of $[x, y]$, r_i is a ramification point, and $r_i \neq r_j$ if $i \neq j$. Let L denote the set to which r belongs if and only if r is r_i for some i and r_i is in $[x, p]$, and let R denote the set to which r belongs if and only if r is r_i for some i and r_i is in $[p, y]$. Then one of L and R, say L, contains an infinite subsequence $r_{n_1}, r_{n_2}, r_{n_3}, \ldots$ of $r_1, r_2, r_3, \ldots$ and $q_{n_1}, q_{n_2}, q_{n_3}, \ldots$ is a sequence which converges to p no two elements of which are in the same arc with p. Then p is a 2-ramification point and the lemma is proved.

LEMMA 2.3. *If p is a limit point of the set of terminal points of a dendroid M and p is not a terminal point of M, then p is either a ramification point or a 2-ramification point of M.*

Proof. Suppose p is neither a terminal point nor a ramification point of M. Then p is a cutpoint of the arc $[x, y]$ for some points x and y in M, and, if q is in M, $[x, y] \cap [p, q]$ is non-degenerate. Since p is a limit point of the set of terminal points of M, there is a sequence $p_1, p_2, p_3, \ldots$ of terminal points of M which converges to p such that neither x nor y is p_i for any i and such that no two points of the sequence are contained in the same arc with p. Thus p is a 2-ramification point.

Let M be a dendroid and let $TR(M)$ be the union of the set of ramification points, the set of terminal points and the set of 2-ramification points of M. Lemmas 2.1, 2.2 and 2.3 immediately give the following

THEOREM 2.1. *If M is a dendroid, $TR(M)$ is closed.*

Example 2.1. Let M denote the union of the set $L_1, L_2, L_3, \ldots$, where L_i is for each i the line segment in the plane from $(0, 0)$ to $(2^{-i}\cos(\pi 2^{-i}),\ 2^{-i}\sin(\pi 2^{-i}))$. The point (0,0) is a limit point of the set of terminal points of M, but M contains no 2-ramification points. Hence, the union of the set of terminal points and the set of 2-ramification points of a dendroid is not necessarily closed.

Example 2.2. The point (0, 0) in Example 1.2 is neither a terminal point nor a ramification point of M but is a limit point of both the set of terminal points and the set of ramification points. Then the union of the set of terminal points and the set of ramification points is not necessarily closed.

Example 2.3. Let M denote the dendroid described in Example 1.2 and let $N = M - ([1, 0) \times 0)$. The point (0, 0) is a limit point of the ramification points of N but is neither a 2-ramification point nor a ramification point of N. Then the set consisting of the ramification points and the 2-ramification points of a dendroid is not necessarily closed.

Examples 2.1, 2.2 and 2.3 illustrate that the set $TR(M)$ for a dendroid M is the smallest subset of a dendroid which contains either the set of terminal points, the set of ramification points or the set of 2-ramification points of M which, in general, is necessarily closed. In addition, as a result of Theorem 1.3, $TR(M)$ is the smallest subset of a dendroid which is necessarily closed and which contains the set of points at which the dendroid is not locally connected.

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