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SOME EXAMPLES OF WEAKLY ASSOCIATIVE LATTICES

BY

E. FRIED AND G. GRÄTZER* (WINNIPEG, CANADA)

1. A *weakly associative lattice*, or *WA-lattice* (called T-lattice in [1] and trellis in [3]) is a non-void set A with two binary operations $\wedge$ and $\vee$ satisfying the following identities:

(1) $x\wedge x = x$ and $x\vee x = x$ (idempotency);

(2) $x\wedge y = y\wedge x$ and $x\vee y = y\vee x$ (commutativity);

(3) $x\wedge(x\vee y) = x$ and $x\vee(x\wedge y) = x$ (absorption identities);

(4) $\big((x\wedge z)\vee(y\wedge z)\big)\vee z = z$ and $\big((x\vee z)\wedge(y\vee z)\big)\wedge z = z$ (weak associative identities).

In a WA-lattice we set $x\leqslant y$ for $x\wedge y = x$ (or, equivalently, for $x\vee y = y$), and then the relation $\leqslant$ satisfies the following rules:

(5) $x\leqslant x$;

(6) $x\leqslant y$ and $y\leqslant x$ imply $x = y$;

(7) for all x and y, there is a z such that $x\leqslant z$ and $y\leqslant z$, and $z\leqslant u$ for all u satisfying $x\leqslant u$ and $y\leqslant u$;

(8) for all x and y, there is a z such that $x\geqslant z$ and $y\geqslant z$, and $z\geqslant u$ for all u satisfying $x\geqslant u$ and $y\geqslant u$.

All of rules (5)-(8) are obvious, $z = x\vee y$ satisfies (7) and $z = x\wedge y$ satisfies (8). Rules (5)-(8) give an alternative axiom system, in terms of $\leqslant$, of a WA-lattice.

Note that (4) takes the following forms:

$x\leqslant z$ and $y\leqslant z$ imply that $x\vee y\leqslant z$, and dually;

$x\leqslant z$ and $y\leqslant z$ imply that $(x\vee y)\vee z = x\vee(y\vee z)$, and dually.

The latter shows that (4) is a weaker form of associativity.

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Every lattice is, of course, a WA-lattice. More importantly, every tournament can be regarded as a WA-lattice (see [1]). A *tournament* is a non-void set A with a binary relation $<$ satisfying the following rule:

(9) for all $x, y \in A$ exactly one of the following three possibilities holds: $x < y$, $x = y$, $y < x$.

Setting $x \leqslant y$ for $x < y$ or $x = y$, it is obvious that a tournament satisfies (5)-(8).

Tournaments play the role for WA-lattices as chains (fully ordered sets) for lattices. Therefore, any identity to be used to define a class of WA-lattices should be first tested whether it holds for tournaments.

A case in point is the distributive identity

(D) $$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

To see that this fails for tournaments take the tournament

$$Z = \{0, 1, 2\},\ 0 < 1,\ 1 < 2,\ 2 < 0.$$

Then

$$0 \wedge (1 \vee 2) = 0 \wedge 2 = 2, \qquad (0 \wedge 1) \vee (0 \wedge 2) = 0 \vee 2 = 0,$$

and so (D) fails in Z. Worse than that, (D) forces associativity. To see this observe that a WA-lattice is a lattice if and only if $\leqslant$ is transitive. Now assume (D) and let $a \leqslant b \leqslant c$. Then

$$\begin{aligned} a \wedge c &= a \wedge (b \vee c) \quad \text{by (D)} \\ &= (a \wedge b) \vee (a \wedge c) = a \vee (a \wedge c) = a \quad \text{by (3)}, \end{aligned}$$

and so $a \leqslant c$.

In this note we describe various distributive identities and some variants of isotone identities and show their independence. The last section contains a result showing the importance of one of these identities.

2. The following is a list of all identities we are going to consider in this note. Some of them are written as inequalities and implications for ease of understanding. It is easy to convert them into identities:

($D_\wedge$) $x \wedge (y \vee z) = \big((x \wedge y) \vee (x \wedge z)\big) \wedge (y \vee z)$,

($D_\vee$) $x \vee (y \wedge z) = \big((x \vee y) \wedge (x \vee z)\big) \vee (y \wedge z)$,

($D^*_\wedge$) $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$,

($D^*_\vee$) $x \vee (y \wedge z) \geqslant (x \vee y) \wedge (x \vee z)$,

(I_1) $x \wedge y \leqslant x \vee y$,

(I_2) $y \leqslant z$ implies that $x \wedge y \leqslant x \vee z$,

(I_3) $y \leqslant z$ implies that $(x \vee y) \vee z \leqslant x \vee z$.

It is obvious that ($D_\wedge$) implies ($D^*_\wedge$), and ($D_\vee$) implies ($D^*_\vee$). Observe that (I_1) is a special case of (I_2).

THEOREM 1. *All seven identities listed above hold in any tournament. The first four* (*the* D-*s*) *define distributivity of a lattice. The last three* (*the* I-*s*) *hold in any lattice.*

Proof. Three elements of a tournament either form a chain or are contained in a subalgebra isomorphic to Z. In the first case they are in a distributive lattice, so the D-s hold. In the latter case we have to check that the D-s hold in Z, this was done in [2] (see also Lemma 2).

Identity (I_1) holds in any lattice. If $y \leqslant z$, then $x \wedge y \leqslant y \leqslant z \leqslant x \vee z$, therefore ($I_2$) holds in any lattice. Finally, if $y \leqslant z$, then $(x \vee y) \vee z = x \vee (y \vee z) = x \vee z$ provided that $\vee$ is associative; hence (I_3) holds in any lattice. This completes the proof.

3. The verifications of ($D_\wedge$) and ($D_\vee$) will be facilitated by the following observation (the elements e and f are called *comparable* if $e \leqslant f$ or $f \leqslant e$):

LEMMA 2. *Let A be a WA-lattice and $a, b, c \in A$. If at least two of the pairs $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$ are comparable, then* ($D_\wedge$) *and* ($D_\vee$) *hold for a, b* and *c.*

Proof. For reasons of symmetry and duality, it is sufficient to verify that

$$a \vee (b \wedge c) = \big((a \vee b) \wedge (a \vee c)\big) \vee (b \wedge c). \tag{10}$$

If $a \geqslant b$, then $(a \vee b) \wedge (a \vee c) = a \wedge (a \vee c) = a$ by (3), proving (10). If $a \geqslant c$, the same reasoning works.

If $a \leqslant b$ and $a \leqslant c$, then the right-hand side of (10) is $(b \wedge c) \vee (b \wedge c) = b \wedge c$ which equals $a \vee (b \wedge c)$ by (4).

Excluding the possibilities discussed above, we are left with $a \leqslant b$ or $a \leqslant c$, and $b \leqslant c$ or $c \leqslant b$. For reason of symmetry we can assume that $b \leqslant c$, in which case (10) takes the following form:

$$a \vee b = \big((a \vee b) \wedge (a \vee c)\big) \vee b. \tag{11}$$

Now if $a \leqslant b$, then we get $b = \big(b \wedge (a \vee c)\big) \vee b$, which holds by (3). Finally, if $a \leqslant c$, then (11) becomes

$$a \vee b = \big((a \vee b) \wedge c\big) \vee b. \tag{12}$$

Since $a \leqslant c$ and $b \leqslant c$, (4) yields $a \vee b \leqslant c$, hence $(a \vee b) \wedge c = a \vee b$. By (4), $(a \vee b) \vee b = a \vee b$, proving (12). This completes the proof of Lemma 2.

Now we are ready to present our two examples.

Let $T_1 = \{o_1, o_2, i_1, i_2\}$ be a tournament and $A_1 = T_1 \cup \{a, b, c\}$. For $x, y \in T_1$ let $x \leqslant y$ in A_1 iff $x \leqslant y$ in T_1. We define $\leqslant$ on A_1 as follows:

(13) $a < b$;

(14) $a < o_1 < x < i_1 < a$ for $x = b$ and $x = c$;

(15) $b < o_2 < x < i_2 < b$ for $x = a$ and $x = c$.

This is shown on Fig. 1 with $x \rightarrow y$ representing $x < y$. Recall that the arrows in T_1 are as yet unspecified.

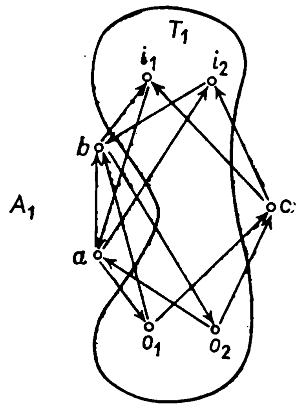

Fig. 1

Observe that all pairs of elements are comparable in A_1 with the exception of $\{a, c\}$ and $\{b, c\}$. Therefore, $x \wedge y$ and $x \vee y$ exist for all pairs $\{x, y\}$ with the possible exceptions of $\{a, c\}$ and $\{b, c\}$. However, both of these pairs have exactly one common upper bound and lower bound, respectively, and so

(16) $$b \wedge c = o_1 \quad \text{and} \quad b \vee c = i_1,$$

(17) $$a \wedge c = o_2 \quad \text{and} \quad a \vee c = i_2.$$

We conclude that A_1 is a WA-lattice.

THEOREM 3. $(D_\wedge)$ *does not imply* $(D_\vee)$. *In fact,* $(D_\wedge)$ *does not imply* $(D_\vee)$ *even in the presence of* $(D_\vee^*)$ *and* (I_1).

Proof. Let us investigate the distributive identities in A_1. By Lemma 2, all distributive identities hold in A_1 under all substitutions excepting a, b, c and its permutations. For $\{y, z\} = \{b, c\}$, $(D_\wedge)$ (and therefore $(D_\wedge^*)$) holds; similarly, for $\{y, z\} = \{a, c\}$, $(D_\vee)$ and $(D_\vee^*)$ hold. For $\{y, z\} = \{a, c\}$, $(D_\wedge^*)$ and also $(D_\wedge)$, is equivalent to $i_2 < o_1$ in T_1. Similarly, for $\{y, z\} = \{b, c\}$, $(D_\vee^*)$ and also $(D_\vee)$, is equivalent to $i_2 < o_1$ in T_1. Finally, for $\{y, z\} = \{a, b\}$, $(D_\wedge^*)$ holds while $(D_\wedge)$ is equivalent to

(18) $$o_1 = b \wedge (o_1 \vee o_2).$$

Dually, for $\{y, z\} = \{a, b\}$, $(D^*_\vee)$ always holds while $(D_\vee)$ holds iff

$$i_2 = a\vee(i_1\wedge i_2). \tag{19}$$

Thus if T_1 is a tournament in which $i_2 < o_1 < o_2$ and $i_2 < i_1$, then (18) holds and (19) fails to hold in A_1, and so A_1 becomes an example in which $(D_\wedge)$ and $(D^*_\vee)$ hold, but $(D_\vee)$ fails to hold.

It is easy to see that (I_1) holds in A_1 iff $o_1 < i_1$ and $o_2 < i_2$. We can assume these to hold in T_1 in addition to the previous relations, so A_1 will also satisfy (I_1). This completes the proof of Theorem 3.

Observe that if we further assume that $i_1 < o_2$, then (I_2) fails in A_1: $a \leqslant b$ does not imply that $a\wedge c \leqslant b\vee c$.

The next example shows that $(D_\wedge)$ does not even imply $(D^*_\vee)$.

THEOREM 4. *$(D_\wedge)$ does not imply $(D^*_\vee)$ even in the presence of (I_2).*

Proof. Let T_2 be a tournament on the six elements o_1, o_2, o_3, i_1, i_2 and i_3. Let $A_2 = T_2\cup\{a_1, a_2, a_3\}$ be a nine element set. We impose on A_2 the relations of T_2 and

$$i_1 < a_1 < o_1, \quad o_2 < a_1 < i_2, \quad o_3 < a_1 < i_3, \tag{20}$$

and the relations for a_2 and a_3 can be derived from (20) by adding 1 and 2, respectively, to the indices (modulo 3), see Fig. 2.

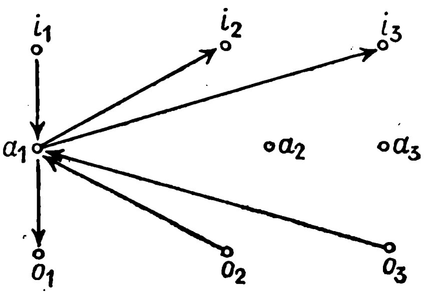

Fig. 2. The relations of (20)

To show that A_2 is a WA-lattice it is sufficient to compute $x\wedge y$ and $x\vee y$ for x and y incomparable; therefore, we can assume that $x \neq y$ and $x, y \in\{a_1, a_2, a_3\}$. For instance, a_1 has three upper bounds o_1, i_2 and i_3; a_2 has three upper bounds o_2, i_3 and i_1, hence $a_1\vee a_2 = i_3$. Similarly, $a_1\wedge a_2 = o_3$. The rest follows by cyclic permutations of the indices.

By Lemma 2 again, it suffices to check the distributive identities for $\{x, y, z\} = \{a_1, a_2, a_3\}$.

For instance,

$$a_1\wedge(a_2\vee a_3) = a_1\wedge i_1 = i_1,$$

$$\big((a_1\wedge a_2)\vee(a_1\wedge a_3)\big)\wedge(a_2\vee a_3) = (o_3\vee o_2)\wedge i_1,$$

thus $(D_\wedge)$ holds iff

(21) $$i_1 \leqslant o_2 \vee o_3, \quad i_2 \leqslant o_1 \vee o_3, \quad i_3 \leqslant o_1 \vee o_2.$$

Similarly, $(D^*_\vee)$ fails if

(22) $$o_3 < i_1 \wedge i_2.$$

So choose the relation in T_2 as follows:

(23) $$o_1 < o_2 < o_3 < o_1, \quad i_3 < o_2, \quad i_2 < o_1, \quad i_1 < o_3,$$

(24) $$o_3 < i_2 < i_1.$$

Then (23) implies (21), and (24) implies (22), therefore, A_2 satisfies $(D_\wedge)$ but not $(D^*_\vee)$.

Finally, we assume that in T_2 we have

(25) $$o_1 < i_1, \quad o_2 < i_2 \quad \text{and} \quad o_3 < i_3.$$

We claim that A_2 satisfies (I_2). Indeed, let $\{x, y, z\} \subseteq A_2$. If $y = z$, then (I_2) reduces to (I_1) which holds by (25). If $\{x, y, z\}$ is a tournament, then (I_2) holds by Theorem 1. Finally, if $y < z$ and $\{x, y, z\}$ is not a tournament, then, say, $y = a_1$, $z = a_2$, and $y \in \{o_1, i_2, i_3\}$; in these cases (I_2) holds. This completes the proof of Theorem 4.

4. For any WA-lattice A there is a maximal lattice homomorphic image $\bar{A}$ under a homomorphism $\varphi: a \to \bar{a}$.

It was observed in [2] that, for WA-lattices in the equational class generated by Z, the map φ can be described as follows:

A sequence $a_0, \ldots, a_{n-1}$ of elements of A is called a *cycle* if $a_0 < a_1 < \ldots < a_{n-1} < a_0$. Then, for $a, b \in A$, $a \neq b$, $a\varphi = b\varphi$ if and only if there exists a cycle containing a and b.

THEOREM 5. *The above-mentioned property of φ holds in any WA-lattice satisfying* (I_3) *and its dual.*

Proof. Let Θ be a binary relation on A defined by the following rule: $a \equiv b(\Theta)$ iff $a = b$ or there is a cycle of A containing a and b. Obviously, Θ is an equivalence relation (for the transitivity of Θ observe that the set union of two cycles can be made into a cycle again since the elements of a cycle are not necessarily distinct).

Let $a \equiv b(\Theta)$ and $c \in A$. We want to show that $a \vee c \equiv b \vee c(\Theta)$ and $a \wedge c \equiv b \wedge c(\Theta)$. These are obvious if $a = b$. Now, let $a \neq b$. Thus there is a cycle $d_0, \ldots, d_{n-1}$ containing a and b. Consider the sequence

(26) $$d_0 \vee c,\ (d_0 \vee c) \vee d_1,\ d_1 \vee c,\ \ldots,\ d_{n-1} \vee c,\ (d_{n-1} \vee c) \vee d_0.$$

Then $d_0 \vee c \leqslant (d_0 \vee c) \vee d_1$ by (4), $(d_0 \vee c) \vee d_1 \leqslant d_1 \vee c$ by (I_3), and so on. Therefore, dropping the repetitions in (26) will yield a cycle containing $a \vee c$ and $b \vee c$. $a \wedge c \equiv b \wedge c(\Theta)$ is verified similarly.

It is obvious that A/Θ is a maximal lattice homomorphic image of A.

In Fig. 3 there is shown that the conclusion of Theorem 5 does not hold in any WA-lattice. In A_3, there are no cycles, however, $|\vec{A}_3| = 1$.

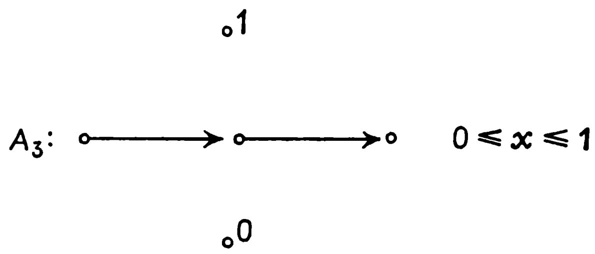

Fig. 3

In the WA-lattice of Fig. 4 the conclusion of Theorem 5 holds but (I_3) fails.

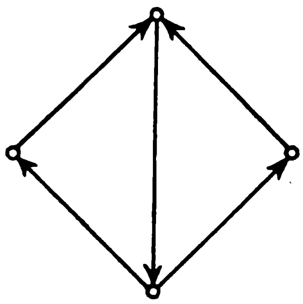

Fig. 4

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THE UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA

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