COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

SOME RECENT RESULTS ON THE LAW OF THE ITERATED LOGARITHM FOR BANACH SPACE VALUED RANDOM VARIABLES

BY

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1. Introduction. For the moment let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) real-valued random variables such that $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 = \sigma^2 > 0$, and as usual write $S_n = X_1 + X_2 + \ldots + X_n$ $(n \geqslant 1)$. In this setting the law of the iterated logarithm (LIL) gives rather precise information on the fluctuations of the sequence $\{S_n\}$ in that it asserts

$$(1.1) \qquad \mathrm{P}\left\{\omega: \lim_n d\left(\frac{S_n(\omega)}{a_n}, [-\sigma, \sigma]\right) = 0\right\} = 1$$

and

$$(1.2) \qquad \mathrm{P}\left\{\omega: C\left(\left\{\frac{S_n(\omega)}{a_n}: n \geqslant 1\right\}\right) = [-\sigma, \sigma]\right\} = 1,$$

where

$$d(x, A) = \inf_{y \in A} |x - y|,$$

$C(\{b_n\})$ stands for all limit points of the sequence $\{b_n\}$, and throughout the paper* we assume that

$$a_n = \begin{cases} \sqrt{2n \log\log n} & \text{for } n \geqslant 3, \\ 1 & \text{otherwise.} \end{cases}$$

Now assume that B is a real separable Banach space with norm $\|\cdot\|$. A rather natural question to ask is whether the analogues of (1.1) and (1.2) hold if $X_1, X_2, \ldots$ are i.i.d. B-valued random variables such that $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. That is, under these assumptions one might hope to

* This is a survey article presented at a special session of the 1976 Annual Summer Meeting of the Institute of Mathematical Statistics and was supported in part by NSF Grant MPS75-05855.

prove that there exists a bounded symmetric set $K \subseteq B$ such that

$$\mathrm{P}\left\{\lim d\left(\frac{S_n}{a_n}, K\right) = 0\right\} = 1 \tag{1.3}$$

and

$$\mathrm{P}\left\{C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K\right\} = 1, \tag{1.4}$$

where

$$d(x, A) = \inf_{y \in A} \|x-y\|$$

and $C(\{b_n\})$ stands for all limit points of $\{b_n\}$ in B.

If B is a finite-dimensional Banach space, then one can produce a limit set K (determined completely by the covariance matrix of X_1) such that (1.3) and (1.4) hold under the classical assumptions mentioned previously. However, if B is infinite dimensional, then (1.3) and (1.4) will not necessarily hold for all i.i.d. sequences $\{X_k\}$ satisfying the classical moment assumptions $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. This can easily be seen from an interesting example due to Dudley and Strassen [3] involving random variables with values in $C[0, 1]$. This example has been constructed to show that the central limit theorem (CLT) does not always hold for Banach space valued random variables, but applies equally well to the LIL.

In the example of Dudley and Strassen the random variables are actually uniformly bounded with probability one, so any result valid for all i.i.d. sequences in all separable Banach spaces must involve something more than moment conditions. Therefore, the following obvious questions present themselves:

(1) Can one prove the LIL for special sequences of i.i.d. random variables with values in spaces like $C[0, 1]$?

(2) For which infinite-dimensional Banach spaces, if any, does the LIL always hold for i.i.d. random variables under the classical moment assumptions $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$?

(3) Does a "general result" hold for all real separable Banach spaces B and for all i.i.d. B-valued random variables satisfying the classical moment assumptions?

Before we answer these questions we first turn to the construction of the limit set K involved in the LIL.

2. The limit set K. Let μ denote a Borel probability measure on B such that

$$\int_B \|x\|^2 d\mu(x) < \infty \quad \text{and} \quad \int_B x\, d\mu(x) = 0.$$

Let S denote the linear operator from B^* to B defined by the Bochner integral

$$(2.1)\qquad Sf = \int_B x f(x)\, d\mu(x) \qquad (f \in B^*).$$

Let H_μ denote the completion of the range of S with respect to the norm obtained from the inner product

$$(2.2)\qquad (Sf, Sg)_\mu = \int_B f(x) g(x)\, d\mu(x).$$

Then, it is fairly easy to see that the following propositions hold:

(i) H_μ can be realized as a subset of B and the identity map $i\colon H_\mu \to B$ is continuous.

In fact, for $x \in H_\mu$ we have

$$(2.3)\qquad \|x\| \leqslant \Big(\int_B \|y\|^2 d\mu(y)\Big)^{1/2} \|x\|_\mu .$$

(ii) If $\Gamma\colon B^* \to H_\mu^*$ is the linear map obtained by restricting an element of B^* to the subspace H_μ of B, and if we identify H_μ^* and H_μ in the usual way, then $\Gamma = S$.

(iii) If K is the unit ball of H_μ, then K is a norm compact, symmetric convex set in B. Further, for each $f \in B^*$ we have

$$(2.4)\qquad \sup_{x\in K} f(x) = \Big\{\int_B [f(y)]^2 d\mu(y)\Big\}^{1/2}.$$

(iv) H_μ is uniquely determined by the covariance operator

$$T(f, g) = \int_B f(x) g(x)\, d\mu(x) \qquad (f, g \in B^*).$$

The proofs of (i), (ii), (iii), and (iv) are fairly elementary, and appear in [6] or [7].

The point of the above construction is that the limiting set K in our results can be taken to be the unit ball of H_μ, where $\mu = \mathscr{L}(X_1)$. Here, of course, $\mathscr{L}(X_1)$ denotes the common distribution induced on B by the i.i.d. random variables $\{X_k\}$.

3. The LIL for B-valued random variables. A partial answer to question (3) is our first theorem which is given in [6].

THEOREM A. *Let* $X_1, X_2, \ldots$ *be i.i.d. B-valued random variables such that* $\mathrm{E}X_1 = 0$ *and* $\mathrm{E}\|X_1\|^2 < \infty$. *Then*

(I) *If K denotes the unit ball of $H_{\mathscr{L}(X_1)}$, then K is a compact symmetric convex subset of B such that*

$$(3.1)\qquad \mathrm{P}\Big\{C\Big(\Big\{\frac{S_n}{a_n}\colon n \geqslant 1\Big\}\Big) \nsubseteq K\Big\} = 0.$$

(II) *In addition, if K is as in* (I), *then*

$$\mathrm{P}\left\{\lim_n d\left(\frac{S_n}{a_n}, K\right) = 0\right\} = 1 \tag{3.2}$$

and

$$\mathrm{P}\left\{C\left(\left\{\frac{S_n}{a_n}:\ n \geqslant 1\right\}\right) = K\right\} = 1 \tag{3.3}$$

iff

$$\mathrm{P}\left\{\left\{\frac{S_n}{a_n}:\ n \geqslant 1\right\} \text{ is conditionally compact in } B\right\} = 1. \tag{3.4}$$

One interesting aspect of Theorem A is (3.1) which implies that all clustering must be in the limit set K constructed in Section 2. Furthermore, it is easy to see that (3.4) and (3.1) imply (3.2), and since K is compact, (3.2) implies (3.4). Hence, what is really required to establish Theorem A is (3.1), and that (3.2) (or (3.4)) implies (3.3). That (3.3) always follows from (3.2) (or (3.4)) is useful to know, since previous to Theorem A this sort of result was often the most difficult aspect of the LIL.

Of course, the main difficulty with Theorem A is in establishing (3.2) or (3.4). A more useful answer to question (3) will follow later (see Theorem 4.1), but first we turn to a brief description of earlier results in the area. Theorems B and C provide answers to question (1) and Theorems D and E to question (2).

THEOREM B (LePage [11]). *If $X_1, X_2, \ldots$ are independent B-valued random variables such that $\mathscr{L}(X_k) = \mu$ $(k \geqslant 1)$, where μ is a mean zero Gaussian measure on B, then* (3.2) *and* (3.3) *hold with K the unit ball of H_μ.*

THEOREM C (Kuelbs [6]). *If $X_1, X_2, \ldots$ are i.i.d. $C[0,1]$-valued such that $\mathrm{E}X_1(t) = 0$ $(0 \leqslant t \leqslant 1)$, $\mathrm{E}\|X_1\|_\infty^2 < \infty$, and $\{X_1(t):\ 0 \leqslant t \leqslant 1\}$ is a martingale in t, then* (3.2) *and* (3.3) *hold with K the unit ball of $H_{\mathscr{L}(X_1)}$.*

Remark. If $R(s,t) = \mathrm{E}X_1(s)X_1(t)$ $(0 \leqslant s, t \leqslant 1)$, then $H_{\mathscr{L}(X_1)}$ equals the reproducing kernel Hilbert space H_R generated by the covariance kernel R.

THEOREM D (Kuelbs [6]). *Let $X_1, X_2, \ldots$ be i.i.d. Hilbert space valued random variables such that $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. Then* (3.2) *and* (3.3) *hold with K equal to the unit ball of $H_{\mathscr{L}(X_1)}$.*

Remark. Theorem D also holds (hence question (2) has an affirmative answer) if $X_1, X_2, \ldots$ satisfy the classical moment conditions and have range B, where the norm on B is sufficiently smooth. For example, see [5] and [6] for such results.

After these earlier results were established it became clear that the LIL should also hold under the classical moment conditions provided the random variables take values in a Banach space of type 2. Such a result seemed most plausible due to parallel results for the CLT in spaces of type 2, and was proved by Pisier in [14]. Pisier's result is our next theorem and it provides a generalization of Theorem D as well as those results for Banach spaces with a smooth norm as all such spaces are of type 2. First, however, recall that a Banach space is of *type* 2 if there exists an absolute constant A such that

$$\mathrm{E}\|X_1+X_2+\ldots+X_n\|^2 \leqslant A \sum_{j=1}^{n} \mathrm{E}\|X_j\|^2$$

for all independent mean zero random variables $X_1, X_2, \ldots, X_n$ and all $n \geqslant 1$.

THEOREM E (Pisier [14]). *Let B denote a Banach space of type* 2 *and assume that $X_1, X_2, \ldots$ are i.i.d. B-valued random variables such that $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. Then* (3.2) *and* (3.3) *hold with K equal to the unit ball of $H_{\mathscr{L}(X_1)}$.*

4. A more complete answer to question (3). As was mentioned previously, the results regarding the LIL parallel those for the CLT for Banach space valued random variables. In fact, Pisier proved in [15] that if $\{X_k\}$ is an i.i.d. sequence of B-valued random variables such that $\mathrm{E}X_1 = 0$, $\mathrm{E}\|X_1\|^2 < \infty$, and $\{S_n/\sqrt{n}\colon n \geqslant 1\}$ converges weakly to a mean zero Gaussian measure, then $\{X_k\}$ satisfies the LIL, i.e. (3.2) and (3.3) hold. Our next result yields this as a simple corollary.

If (M, d) is a metric space, $\{x_n\}$ a sequence of points in M, and $A \subseteq M$, we will use the notation $\{x_n\} \twoheadrightarrow A$ for both

$$\lim_n d(x_n, A) = 0 \quad \text{and} \quad C(\{x_n\}) = A.$$

THEOREM 4.1 (Kuelbs [8]). *Let $X_1, X_2, \ldots$ be i.i.d. B-valued random variables such that $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. If K is the unit ball of $H_{\mathscr{L}(X_1)}$, then the following conditions are equivalent*:

(i) $\mathrm{P}\{\{S_n/a_n\colon n \geqslant 1\} \twoheadrightarrow K\} = 1$,

(ii) $\mathrm{E}\|S_n\| = o(a_n)$,

(iii) $S_n/a_n \to 0$ *in probability.*

An immediate corollary to Theorem 4.1 is the following result. Furthermore, it is easy to see that Theorems B, C, D, and E are immediate consequences of Theorem 4.1.

COROLLARY 4.1 (Pisier [15]). *Let $X_1, X_2, \ldots$ be i.i.d. B-valued random variables such that $\mathrm{E}X_1 = 0$ and $\mathrm{E}\|X_1\|^2 < \infty$. If $\{S_n/\sqrt{n}\colon n \geqslant 1\}$ is stochastically bounded, then*

$$\mathrm{P}\left\{\left\{\frac{S_n}{a_n}\colon n \geqslant 1\right\} \twoheadrightarrow K\right\} = 1,$$

where K is the unit ball of $H_{\mathscr{L}(X_1)}$.

5. An exponential moment with applications to some estimation problems. The techniques used to establish Theorem 4.1 can be applied to calculate moments of random variables of the form

$$M = \sup_n \frac{\|X_1+X_2+\ldots+X_n\|}{a_n}. \tag{5.1}$$

We know that if $\{X_k\}$ satisfies the LIL, then $\mathrm{P}(M<\infty)=1$ and

$$\limsup_n \frac{\|X_1+X_2+\ldots+X_n\|}{a_n}$$

is a finite constant. Hence one might guess that M should have exponential moments whenever the individual X_k's have them. This is the case, and we next describe a result in this area. After that we indicate some applications to two classical estimation problems in statistics.

However, for the applications we need some additional terminology. Let B denote a real vector space, $\mathscr{B}$ a σ-algebra of subsets of B, and $\|\cdot\|$ a semi-norm on B. We say that the triple $(B, \mathscr{B}, \|\cdot\|)$ is a *linear measurable space* if

(i) addition and scalar multiplication are $\mathscr{B}$-measurable operations on B,

(ii) for all $t \geqslant 0$, $\{x \in B\colon \|x\| \leqslant t\}$ is $\mathscr{B}$-measurable,

(iii) there exists a subset F of the $\mathscr{B}$-measurable linear functionals on B such that

$$\|x\| = \sup_{f\in F} |f(x)| \qquad (x \in B). \tag{5.2}$$

Examples of linear measurable spaces are readily available in probability theory and, of course, include the situation where B is a real separable Banach space, $\mathscr{B}$ denotes the Borel subsets of B, and $\|\cdot\|$ is the norm on B. Another important example consists of $B = D(\boldsymbol{R}^1)$, where $D(\boldsymbol{R}^1)$ denotes the real-valued functions on $\boldsymbol{R}^1$ which are right continuous and have left-hand limits throughout $\boldsymbol{R}^1$. In this case, $\mathscr{B}$ consists of the minimal σ-algebra making the maps $x \to x(t)$ $(t \in \boldsymbol{R}^1)$ measurable, and we can use any of the semi-norms

$$\|x\|_T = \sup_{|t|\leqslant T} |x(t)| \qquad (0 \leqslant T \leqslant \infty).$$

That $\big(D(\boldsymbol{R}^1), \mathscr{B}, \|\cdot\|_T\big)$ is actually a measurable linear space follows easily from the fact that an element in $D(\boldsymbol{R}^1)$ is uniquely determined by its values on any fixed countable dense subset of $\boldsymbol{R}^1$ and then emphasizing the ideas of [4].

A simplified version of the main theorem regarding exponential moments of random variables of the form given in (5.1) is the following:

THEOREM 5.1 (Kuelbs [9]). *Let* $(B, \mathscr{B}, \|\cdot\|)$ *be a linear measurable space and assume that* $X_1, X_2, \ldots$ *are i.i.d.* $(B, \mathscr{B})$*-valued random variables such that*

(a) $\mathrm{E}f(X_1) = 0$ *for all* $f \in F$,

(b) $\mathrm{E}(\exp\{\beta\|X_1\|^2\}) < \infty$ *for some* $\beta > 0$,

(c) $\{S_n/\sqrt{n}\colon n \geqslant 1\}$ *is bounded in probability, where*

$$S_n = \sum_{j=1}^{n} X_j.$$

Then there exists $\beta_0 > 0$ *such that* $\beta \leqslant \beta_0$ *implies*

$$\mathrm{E}\left(\exp\left\{\beta \sup_n \left\|\frac{S_n}{a_n}\right\|^2\right\}\right) < \infty. \tag{5.3}$$

Furthermore, if (b) *holds for all* $\beta > 0$, *then* (5.3) *also holds for all* $\beta > 0$.

Theorem 5.1 applies easily to two basic estimation problems in statistics, and it is this that we turn to now.

The first problem is the use of the empirical distribution function based on an i.i.d. sequence $\{X_k\}$ to estimate the common distribution function of the X_k's.

That is, if $X_1, X_2, \ldots$ are independent real-valued random variables with common distribution function

$$F(x) = \mathrm{P}(X_1 \leqslant x) \qquad (x \in \boldsymbol{R}^1),$$

then the empirical distribution function is given by

$$\mathscr{E}_n(x) = \sum_{j=1}^{n} \frac{1_{(-\infty, x]}(X_j)}{n} \qquad (x \in \boldsymbol{R}^1, n \geqslant 1).$$

Of course, $\{\mathscr{E}_n(x)\colon x \in \boldsymbol{R}^1,\ n \geqslant 1\}$ is a sequence of stochastic processes indexed by $\boldsymbol{R}^1$ and for each $x \in \boldsymbol{R}^1$ we have $\mathrm{E}(\mathscr{E}_n(x)) = F(x)$.

By the law of large numbers we thus have with probability one

$$\lim_n |\mathscr{E}_n(x) - F(x)| = 0 \qquad (x \in \boldsymbol{R}^1),$$

and if F is continuous, it easy to see that

$$\lim_n \sup_{x \in \boldsymbol{R}^1} |\mathscr{E}_n(x) - F(x)| = 0.$$

The importance of the empirical distribution is due to the fact that it allows us to use observations of the i.i.d. sequence $\{X_k\}$ to estimate F. A result of Chung [2] gives us a rate of convergence of $\mathscr{E}_n$ to F when F is continuous, as it asserts that if

$$D_n = \sup_x |\mathscr{E}_n(x) - F(x)|,$$

then

$$\mathrm{P}\left\{\overline{\lim_n} \sqrt{\frac{n}{2\log\log n}}\, D_n = \frac{1}{2}\right\} = 1.$$

Hence, with probability one there is a finite function $C(\omega)$ such that

$$D_n(\omega) \leqslant C(\omega)\sqrt{\frac{\log\log n}{n}}.$$

Of course, for practical purposes we would like $C(\omega)$ uniformly bounded, but this is not the case. However, $C(\omega)$ has exponential moments. That is, write

$$M = \sup_n \sqrt{\frac{n}{2\log\log n}}\, D_n.$$

Then we can assume that $C \leqslant M$, and if

$$Y_j(x, \omega) = 1_{(-\infty, x]}(X_j(\omega)) - F(x) \quad (j \geqslant 1, -\infty < x < \infty),$$

then it is known that $\sum_{j=1}^{n} \{Y_j/\sqrt{n}\colon n \geqslant 1\}$ is stochastically bounded in $(D(R^1), \mathscr{B}, \|\cdot\|_\infty)$. Hence an easy application of Theorem 5.1 gives

THEOREM 5.2 (Kuelbs [9]). *Let $X_1, X_2, \ldots$ be independent real-valued random variables with common distribution function F. Then, for all $\beta > 0$*

$$\mathrm{E}(\exp\{\beta M^2\}) < \infty,$$

where M is defined as above.

Now we turn to our second application. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of real-valued variables with common probability density function $f(x)$ $(x \in R^1)$. A problem of considerable practical importance and also of theoretical interest is the estimation of $f(x)$ by some statistic based, of course, on the observed sequence $\{X_k\}$. Such statistics are frequently called *empirical density functions.*

There is a great deal of literature on this subject and we urge the reader to examine [1], [12], [13], [17], and [18] for background as well as for further references. The paper [1] and some recent work by Révész in [16] deals with the problem of determining limit theorems for the empirical density function, but here we only consider the more classical problem of obtaining a uniform estimate for the density.

The uniform estimates which we obtain are as good or, in most cases, better than those available in the literature, and our method of proof also yields the additional new fact that the estimates when centered at their mean have exponential moments. Moreover, we can also handle the situation where there is some "noise" in the observations $\{X_k\}$, but we present only the most basic result here.

The estimates which we form to approximate $f(x)$ follow those used extensively in regard to this problem. That is, given a weight function K and a sequence $\{h_n\}$ of positive numbers such that

$$\lim_n h_n = 0$$

we write

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h_n}\right) \quad (x \in \boldsymbol{R}^1). \tag{5.4}$$

Then $\{f_n\colon n \geqslant 1\}$ is a sequence of stochastic processes (statistics) on $\boldsymbol{R}^1$ depending on the observed sequence $\{X_k\}$, and we use them to estimate the probability density function $f(x)$.

If $h(x)$ is any real-valued function on $\boldsymbol{R}^1$, we define the bounded Lipschitz norm of h to be

$$\|h\|_{\mathrm{BL}} = \sup_{x\in \boldsymbol{R}^1}|h(x)| + \sup_{x\neq y}\frac{|h(x)-x(y)|}{|x-y|}.$$

THEOREM 5.3 (Kuelbs [10]). *Let $\{X_k\}$ be an i.i.d. sequence of $\boldsymbol{R}^1$-valued random variables having a probability density function f such that $\|f\|_{\mathrm{BL}} < \infty$. Let $\{f_n\}$ be the sequence of estimators defined by* (5.4) *and assume that*

(1) *$\{h_n\}$ is a sequence of positive numbers converging to zero;*

(2) *the kernel K is a probability density function defined on $\boldsymbol{R}^1$ such that*

(a) *K is right continuous and of bounded variation on $\boldsymbol{R}^1$,*

(b) $\int_{\boldsymbol{R}^1} |u| K(u)\,du < \infty$.

Then

(i) $$\sup_{x\in \boldsymbol{R}^1}|f_n(x) - \mathrm{E}f_n(x)| = O\left(\sqrt{\frac{\log\log n}{n}}\Big/ h_n\right);$$

(ii) $$M \equiv \sup_n \sup_{x\in\boldsymbol{R}^1}|f_n(x)-\mathrm{E}f_n(x)|\sqrt{\frac{n}{\log\log n}}\,h_n$$

is a random variable such that

$$\mathrm{E}(\exp\{\beta M^2\}) < \infty \quad \textit{for all } \beta > 0;$$

(iii) *for any positive sequence* $\{b_n\}$ *such that* $b_n \to 0$ *as* $n \to \infty$ *we have*

$$\lim_n \mathrm{E}\left\{\Phi\left(\sqrt{\frac{n}{\log\log n}}\, h_n b_n \sup_{x\in \boldsymbol{R}^1} |f_n(x) - \mathrm{E}f_n(x)|\right)\right\} = 0,$$

Φ *being any non-negative function on* $[0, \infty)$ *such that* $\Phi(0) = 0$,

$$\lim_{t\downarrow 0} \Phi(t) = 0, \quad \textit{and} \quad \Phi(t) \leqslant \exp\{\beta t^2\} \quad \textit{for } t > 0 \textit{ and some } \beta > 0;$$

(iv) *for* $h_n = n^{-1/4}$ *we have*

$$\sup_{x\in \boldsymbol{R}^1} |f_n(x) - f(x)| = O(n^{-1/4}\log\log n);$$

(v) *for* Φ *as in* (iii), *and* $\{h_n\}$ *such that* $n^{-1/2}(\log\log n)^{1/2}/h_n \to 0$ *as* $n \to \infty$ *we have*

$$\lim \mathrm{E}\left\{\Phi\left(\sup_{x\in \boldsymbol{R}^1} |f_n(x) - f(x)|\right)\right\} = 0.$$

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Reçu par la Rédaction le 20. 9. 1976