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THE INVARIANCE PRINCIPLE FOR RANDOM VARIABLES WITH VALUES IN LOCALLY COMPACT GROUPS

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Let G be a locally compact group satisfying the second axiom of countability. Let $D_G = D_G[0,1]$ be the space of all functions f defined on $[0,1]$ with values in G that are right-continuous and have left-hand limits. D_G endowed with the so-called Skorohod topology is a *separable topologically complete space*. Let

$$\{X_j^{(n)}\colon j = 1, \ldots, n;\ n = 1, 2, \ldots\}$$

be an infinitesimal triangular array of symmetric, G-valued, independent and identically distributed random variables. Let us put

$$S_k^{(n)} = X_1^{(n)} X_2^{(n)} \ldots X_k^{(n)}.$$

This (double) sequence defines a sequence of G-valued stochastic processes $\{\xi_n\}$ with the sample paths in D_G:

$$\xi_n(t) = S_{[nt]}^{(n)}.$$

Assume that $\xi_n(1)$ converges in distribution to a G-valued Gaussian random variable.

Is the sequence $\{\xi_n\}$ (considered as a sequence of random elements with values in D_G) convergent in distribution to a stochastic process with continuous sample paths?

The positive answer to this question was given by Donsker in [3] for G being the real line, and by Byczkowski in [2] for G being an LCA group (satisfying the second axiom of countability).

The aim of this paper* is to extend this result for non-Abelian locally compact groups G. The proof of the invariance principle is based on methods developed in [2]. However, we have to deal with the operator-valued characteristic functions of probability measures and the situation becomes more complicated.

* This paper is based on the author's doctoral thesis written under the supervision of Professor C. Ryll-Nardzewski.

At the end of the paper we consider G-valued homogeneous stochastic processes with independent increments, having continuous sample paths, and we prove that every such process is Gaussian.

1. Preliminaries. Throughout the paper, G will always denote a locally compact group. We will assume that G satisfies the second axiom of countability, unless otherwise stated.

Let ϱ be a left-invariant metric generating the topology of G. It is well known (and not hard to verify) that ϱ is complete.

Let $\|\cdot\|$ denote the distance from the identity e of G: $\|x\| = \varrho(x, e)$.

By $D_G = D_G[0, 1]$ we will denote the space of all functions f defined on $[0, 1]$ with values in G that are right-continuous and have left-hand limits.

Let Λ denote the class of all strictly increasing, continuous mappings of $[0, 1]$ onto itself taking 0 onto 0. For f and g in D_G define $d(f, g)$ to be the infimum of those positive ε for which in Λ there exists a λ such that

$$\sup_t |\lambda t - t| \leqslant \varepsilon \quad \text{and} \quad \sup_t \|(f(t))^{-1} g(\lambda t)\| \leqslant \varepsilon.$$

The space D_G is a separable, topologically complete metric space with the topology generated by the metric d (the so-called Skorohod topology).

A mapping ξ defined on a probability space $(\Omega, \mathfrak{S}, P)$ with values in D_G is called a *random element* if it is measurable with respect to $\mathfrak{S}$ and the Borel σ-algebra in D_G.

ξ is a random element (with values in D_G) if and only if $\xi(t)$ is a random variable with values in G (i.e., $\xi(t)$ is measurable with respect to $\mathfrak{S}$ and the Borel σ-algebra in G).

For further information concerning the space D_G as well as characterizations of the uniform tightness of families of probability measures on D_G, the reader is referred to [1] (the arguments used there apply to our general situation without almost any change).

Now, let us remind some basic facts about representations of locally compact groups and about the Fourier transform of probability measures.

Let U be a unitary representation of a locally compact group G, that is, a homomorphism $g \to U(g)$ of G into the group of unitary operators on certain Hilbert space H. The representation U is called (*weakly*) *continuous* if the mapping

$$g \to \langle U(g)x, y\rangle$$

is continuous for every $x, y \in H$. A subspace $L \subset H$ is called an *invariant subspace* of U if $UL \subset L$, that is, $U(g)x \in L$ for every $g \in G$ and every $x \in L$. We say that U is *irreducible* if there exists no proper invariant subspace of U.

Two unitary representations U and V of G are called *equivalent* if there exists a linear isometry T such that

$$U_g T = T V_g \quad \text{for all } g \in G.$$

Let $\mathscr{U}'(G)$ be the set of all equivalence classes of continuous, irreducible unitary representations of G. By $\mathscr{U}(G)$ we will denote a fixed selector of $\mathscr{U}'(G)$.

Now, if μ is a probability measure on G (that is, a positive Borel measure satisfying $\mu(G) = 1$) we can define the Fourier transform $\hat{\mu}$ by the formula

$$\langle \hat{\mu}(U)x, y\rangle = \int_G \langle U(g)x, y\rangle \mu(dg), \qquad U \in \mathscr{U}(G),\ x, y \in H.$$

It is well known that the mapping $\mu \to \hat{\mu}$ is one-to-one and that $(\mu_1 * \mu_2)^{\hat{}} = \hat{\mu}_1\hat{\mu}_2$ (see [5]). Moreover, $\hat{\mu}(U)$ is a self-adjoint operator for every $U \in \mathscr{U}(G)$ if and only if μ is symmetric, that is, if $\mu(E) = \mu(E^{-1})$ for every Borel $E \subset G$.

It is also known that if μ_n is weakly convergent to μ, then

$$\hat{\mu}_n(U)x \to \hat{\mu}(U)x \quad \text{for every } x \in H \text{ and every } U \in \mathscr{U}(G).$$

Conversely, if

$$\langle \hat{\mu}_n(U)x, y\rangle \to \langle \hat{\mu}(U)x, y\rangle \quad \text{for every } x, y \in H \text{ and every } U \in \mathscr{U}(G),$$

then μ_n converges weakly to μ.

Now, a family $(\mu_t)_{t>0}$ is called a *semigroup of probability measures* if $\mu_t * \mu_s = \mu_{t+s}$ for every $t, s > 0$; it is called a *continuous semigroup* if the mapping $t \to \mu_t$ is continuous. $(\mu_t)_{t>0}$ is called *e-continuous* if

$$\lim_{t \to 0+} \mu_t = e$$

(the point measure concentrated at the identity of G). It is known [11] that a semigroup $(\mu_t)_{t>0}$ is continuous if and only if $\lim_{t\to 0+}\mu_t$ exists. If this limit exists, it is the identity of the semigroup (hence idempotent).

A probability measure μ will be called *embeddable* if there exists a continuous semigroup $(\mu_t)_{t>0}$ of probability measures such that $\mu_1 = \mu$.

A probability measure μ is called *Gaussian* if it is embeddable into an e-continuous semigroup $(\mu_t)_{t>0}$ of probability measures having the property

$$\lim_{t \downarrow 0} \frac{1}{t} \mu_t(G \setminus V) = 0 \tag{1}$$

for every open neighbourhood V of the identity of G (see Section 4 in [7]). More precisely, we will call μ a *Gaussian measure with embedding semigroup* $(\mu_t)_{t>0}$.

For an equivalent definition in terms of Lévy-Khintchine representation, see [6] and [10].

2. The invariance principle. Let $\{S_k^{(n)}: k = 1, \ldots, n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued random variables. Let $\nu_k^{(n)}$ denote the distribution of $S_k^{(n)}$.

LEMMA 1. *The following conditions are equivalent:*

(i) $\lim\limits_{h\downarrow 0}\overline{\lim\limits_{n\to\infty}}\max\limits_{k\leqslant nh} P\{\|S_k^{(n)}\| \geqslant \delta\} = 0$ *for every* $\delta > 0$.

(ii) $\lim\limits_{h\downarrow 0}\overline{\lim\limits_{n\to\infty}}\max\limits_{k\leqslant nh} \sigma(\nu_k^{(n)}, e) = 0$, *where σ is a metric generating the topology of weak convergence of probability measures on G.*

(iii) $\lim\limits_{h\downarrow 0}\overline{\lim\limits_{n\to\infty}}\max\limits_{k\leqslant nh} \|\hat{\nu}_k^{(n)}(U)x - x\| = 0$ *for every $x \in H$ and every $U \in \mathscr{U}(G)$.*

Proof. Observe that condition (i) can be stated equivalently:

(i)′ $\mathop{\forall}\limits_{\{h_j\}} (h_j\downarrow 0)\ \mathop{\forall}\limits_{\{n_j\}} (n_j\uparrow\infty)\ \mathop{\forall}\limits_{\{k_j\}} (0 \leqslant k_j \leqslant n_j h_j)\ S_{k_j}^{(n_j)} \to 0$ in probability.

Since the convergence in probability to a constant is equivalent to the convergence in distribution to a point measure, conditions (i)′ and (ii) are equivalent.

Next, it is obvious that (ii) implies (iii).

Finally, assume that (iii) is satisfied. Let $\{h_j\}$, $\{n_j\}$ and $\{k_j\}$ have the properties as in (i)′. Then $\hat{\nu}_{k_j}^{(n_j)}(U)x \to x$ for every $x \in H$ and every $U \in \mathscr{U}(G)$. In virtue of the properties of the Fourier transform we infer that $\nu_{k_j}^{(n_j)}$ converges weakly to e, which is equivalent to (ii).

The next lemma is taken from [2].

LEMMA 2. *Let $\{X_j^{(n)}: j = 1, \ldots, n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued, independent and identically distributed random variables. Let $S_k^{(n)} = X_1^{(n)} X_2^{(n)} \ldots X_k^{(n)}$. Suppose that*

(a) *$S_{[nt]}^{(n)}$ converges weakly for $t \in F$, where F is a dense subset of $[0, 1]$ containing* 1;

(b) *for each $\varepsilon > 0$ and a certain sequence δ_n, $0 < \delta_n < 1$, $\delta_n \to 0$,*

$$\frac{1}{\delta_k}\overline{\lim_n} P\{\|S_{[n\delta_k]}^{(n)}\| > \varepsilon\} \to 0 \quad \textit{as } k \to \infty;$$

(c) *for each $\varepsilon > 0$ there exists $h > 0$ such that*

$$\overline{\lim_n}\max_{1\leqslant r\leqslant nh} P\{\|S_r^{(n)}\| > \varepsilon\} < 1.$$

Let $\xi_n(t) = S_{[nt]}^{(n)}$. Then the sequence of random elements ξ_n converges weakly to a certain random element of D_G with continuous sample paths.

Let a probability measure μ be embedded into a continuous semigroup $(\mu_t)_{t>0}$ of probability measures. We prove that there exists at most one symmetric semigroup $(\mu_t)_{t>0}$ with the properties above. This is an immediate consequence of the following well-known lemma, applied to the Fourier transforms of $(\mu_t)_{t>0}$:

LEMMA 3. *Let $(T_t)_{t>0}$ be a strongly continuous semigroup of self-adjoint operators on a Hilbert space. Then $T_t \geqslant 0$ for every $t > 0$ and*

$$T_t = \int_{\sigma(T_1)} \lambda^t E(d\lambda),$$

where E is the spectral measure of T_1, and λ^t, $t > 0$, is the real power of $\lambda \geqslant 0$.

COROLLARY 1. *Let μ be a symmetric probability measure on G. There exists at most one continuous semigroup $(\mu_t)_{t>0}$ of symmetric probability measures such that $\mu_1 = \mu$.*

Let T be a self-adjoint bounded operator on a Hilbert space H. Let f be a bounded, Borel measurable function defined on the spectrum of T. By $f(T)$ we denote the operator defined by the formula

$$f(T) = \int f(\lambda) E(d\lambda),$$

where E is the spectral measure of T (see [4], X, Section 2). If T is, in addition, non-negative, then by T^a ($a > 0$) we will denote the operator $f(T)$, where $f(\lambda) = \lambda^a$ ($\lambda \geqslant 0$) is the real power of λ.

The next lemma is a consequence of Theorem 2 in [4], X, Section 8.

LEMMA 4. *Let T_n and T be bounded operators on a Hilbert space H. Assume that T_n is non-negative and that T_n converges strongly to T. Let $a_n \to a$, $a_n, a > 0$. Then $T_n^{a_n}$ converges strongly to T^a.*

LEMMA 5. *Let S_n and T be bounded operators on a Hilbert space. Assume that S_n are self-adjoint, $\|S_n\| \leqslant 1$, S_n converges strongly to I (the identity operator) and S_n^n converges strongly to T. Then $T \geqslant 0$ (T is non-negative) and if $k_n/n \to t$ for $t > 0$ and some integers k_n, then $S_n^{k_n}$ converges strongly to T^t.*

Proof. Let $k_n = k_n' + \varepsilon_n$, where k_n' is an even integer and $\varepsilon_n = 0$ or $\varepsilon_n = 1$. Then $S_n^{k_n} = S_n^{k_n'} S_n^{\varepsilon_n}$ and

$$S_n^{k_n'} = \begin{cases} (S_n^n)^{k_n'/n} & \text{if } n \text{ is even,} \\ (S_n^{n+1})^{k_n'/(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

Let us put

$$T_n = \begin{cases} S_n^n & \text{if } n \text{ is even,} \\ S_n^{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

By assumption, T_n is strongly convergent to T. Since T_n are non-negative, T is also non-negative. It follows from Lemma 4 that $S_n^{k_n'} = T_n^{k_n'/n}$ converges strongly to T^t. Since S_n converges strongly to I, we infer that $S_n^{k_n} = S_n^{k_n'} S_n^{\varepsilon_n}$ converges strongly to T^t, which completes the proof.

THEOREM 1. *Let* $\{X_k^{(n)}: k = 1, \ldots, n;\ n = 1, 2, \ldots\}$ *be an infinitesimal triangular array of* G*-valued, symmetric, independent and identically distributed random variables. Let* ν_n *be the distribution of* $X_k^{(n)}$. *Assume that* ν_n^{*n} *converges weakly to* μ. *Then* $\hat{\mu}(U) \geqslant 0$ *for every* $U \in \mathscr{U}(G)$ *and* $\nu_n^{*[nt]}$ *converges weakly to* μ_t, *where* $(\mu_t)_{t>0}$ *is a continuous semigroup of probability measures such that* $\hat{\mu}_t(U) = \int \lambda^t E(d\lambda)$, *and* E *is the spectral measure of* $\hat{\mu}(U)$.

Proof. Let $U \in \mathscr{U}(G)$. Let us write $S_n = \hat{\nu}_n(U)$ and $T = \hat{\mu}(U)$. Since the considered array is symmetric and infinitesimal, $S_n = S_n^*$ and S_n converges strongly to I. By assumption, S_n^n converges strongly to T. Let $k_n = [nt]$ for $t > 0$. Since $k_n/n \to t$, we infer from Lemma 5 that $T \geqslant 0$ and

$$S_n^{[nt]} \text{ converges strongly to } T^t = \int_{\sigma(T)} \lambda^t E(d\lambda),$$

where E is the spectral measure of T. Clearly, $T^t T^s = T^{t+s}$ for $t, s > 0$.

Next, we show that $\{\nu_n^{*[nt]}\}$ is weakly compact for every fixed $t > 0$. To show this, let us write $[nt] = 2m_n + \varepsilon_n$, where $\varepsilon_n = 0$ or $\varepsilon_n = 1$. Since $\nu_n^{*m_n} * \nu_n^{*(n-m_n)}$ converges weakly to μ, we infer from Theorem 2.1, III, in [8] that there exists $\{g_n\}$, $g_n \in G$, such that $\{\nu_n^{*m_n} * g_n\}$ is weakly compact. Then $\{(-g_n) * \nu_n^{*m_n}\}$ is also weakly compact, hence so is

$$\nu_n^{*[nt]} = (\nu_n^{*m_n} * g_n) * \big((-g_n) * \nu_n^{*m_n}\big) * \nu_n^{*\varepsilon_n}.$$

Since we have just shown that $\big(\hat{\nu}_n(U)\big)^{[nt]}$ is strongly convergent to $\hat{\mu}_t(U)$, $(\hat{\mu}_t)$ is the Fourier transform of a probability measure μ_t and $\mu_t * \mu_s = \mu_{t+s}$ for every $t, s > 0$. This completes the proof of the theorem.

Remark 1. Let μ be embedded into a continuous semigroup of probability measures $(\mu_t)_{t>0}$. Then $(\mu_t)_{t>0}$ is e-continuous if and only if $\hat{\mu}_t(U)x$ converges to x for every $U \in \mathscr{U}(G)$ and every $x \in H$.

If $\hat{\mu}_t(U) = \int \lambda^t E(d\lambda)$ for $t > 0$, where E is concentrated on $[0, 1]$, then the condition above is satisfied if and only if for every $U \in \mathscr{U}(G)$ the spectral measure E of $\hat{\mu}(U)$ has no atom at 0. Hence we have

COROLLARY 2. *The following conditions are equivalent*:

(i) μ *is embeddable into an* e*-continuous symmetric semigroup of probability measures.*

(ii) μ *has no non-trivial idempotent factors and there exists an infinitesimal triangular array of symmetric, independent and identically distributed random variables* $\{X_k^{(n)}: k = 1, \ldots, n;\ n = 1, 2, \ldots\}$ *such that* $S_n^{(n)}$ *converges in distribution to* μ.

LEMMA 6. *Let* T_n *and* T *be bounded non-negative operators on a Hilbert space. Assume that* $T_n \leqslant I$ *and that* T_n *converges strongly to* T. *Let* E *be the spectral measure of* T. *If* $E(\{0\}) = 0$, *then*

$$\lim_{h \downarrow 0} \overline{\lim_{n}} \sup_{a \leqslant h} \|T_n^a x - x\| = 0.$$

Proof. Let $0 < a < h$. Then $T_n^h \leqslant T_n^a$ and $I - T_n^a \leqslant I - T_n^h$. Hence

$$\langle (I - T_n^a)x, x\rangle \leqslant \langle (I - T_n^h)x, x\rangle.$$

Since for normal operator A

$$\|A\| = \sup_{\|x\| \leqslant 1} \langle Ax, x\rangle,$$

we obtain

$$\sup_{a \leqslant h} \|T_n^a x - x\| = \|T_n^h x - x\|.$$

Since, by assumption, T_n converges strongly to T, we have

$$\overline{\lim_n} \sup_{a \leqslant h} \|T_n^a x - x\| = \|T^h x - x\|.$$

Moreover, since $E(\{0\}) = 0$, we infer that T^h converges strongly to I as $h \to 0$, hence

$$\lim_{h \downarrow 0} \overline{\lim_n} \sup_{a \leqslant h} \|T_n^a x - x\| = 0,$$

which completes the proof.

LEMMA 7. *Let S_n and T be bounded operators on a Hilbert space. Assume that $\|S_n\| \leqslant 1$, $S_n^* = S_n$, S_n converges strongly to I and that S_n^n converges strongly to T. Let E be the spectral measure of T. If $E(\{0\}) = 0$, then*

$$\lim_{h \downarrow 0} \overline{\lim_n} \max_{k \leqslant nh} \|S_n^k x - x\| = 0.$$

Proof. Let $h \leqslant 1$ be a positive real and let $k_n \leqslant nh$ be an integer. Then

$$S_n^{k_n} = S_n^{k_n'} S_n^{s_n} = T_n^{a_n} S_n^{s_n},$$

where k_n', s_n and T_n have the same meanings as in the proof of Lemma 5 and

$$a_n = \begin{cases} k_n'/n & \text{if } n \text{ is even,} \\ k_n'/(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

Observe that

$$\|S_n^{k_n} x - x\| \leqslant \|T_n^{a_n} x - x\| + \|T_n^{a_n} S_n^{s_n} x - T_n^{a_n} x\| \leqslant \|T_n^{a_n} x - x\| + \|S_n x - x\|.$$

T_n are non-negative and $T_n \leqslant I$, hence, in virtue of Lemma 6, we obtain the conclusion.

LEMMA 8. *Let $(\mu_t)_{t>0}$ be an e-continuous semigroup of symmetric probability measures on G and let $\{X_k^{(n)}\}$ be an infinitesimal triangular array of G-valued, symmetric, independent and identically distributed random variables. Let ν_n be the distribution of $X_j^{(n)}$, $j = 1, 2, \ldots, n$. Assume that ν_n^{*n} converges weakly to μ_1. Then for every $x \in H$ and every $U \in \mathcal{U}(G)$ we have*

$$\lim_{h \downarrow 0} \overline{\lim_n} \max_{k \leqslant nh} \|\hat{\nu}_n(U)^k x - x\| = 0.$$

Proof. Applying Lemma 7 together with Remark 1 to $S_n = \hat{\nu}_n(U)$ and $T = \hat{\mu}_1(U)$ we obtain the conclusion.

THEOREM 2. *Let μ be a symmetric Gaussian measure on G with an e-continuous symmetric embedding semigroup $(\mu_t)_{t>0}$. Let $\{X_k^{(n)}: k = 1, \ldots, n;\ n = 1, 2, \ldots\}$ be an infinitesimal triangular array of symmetric, independent and identically distributed G-valued random variables. Assume that $S_n^{(n)} = X_1^{(n)} X_2^{(n)} \ldots X_n^{(n)}$ converges in distribution to μ. Let $\xi_n(t) = S_{[nt]}^{(n)}$. Then the sequence of D_G-valued random elements ξ_n converges in distribution to a random element W with continuous sample paths.*

Proof. Let ν_n be the distribution of $X_k^{(n)}$, $k = 1, \ldots, n$. From Theorem 1 we infer that $\nu_n^{*[nt]}$ converges weakly to ν_t, where $(\nu_t)_{t>0}$ is a continuous semigroup of probability measures such that $\hat{\nu}_t(U) = \int \lambda^t E(d\lambda)$ (E is the spectral measure of $\hat{\mu}(U)$). By Corollary 1 we obtain $\nu_t = \mu_t$. Thus condition (a) of Lemma 2 is satisfied. Since μ is a Gaussian measure with an embedding semigroup (μ_t), we obtain (1) for every open neighbourhood V of the identity of G. Since $\nu_n^{*[nt]}$ converges weakly to μ_t, we obtain

$$\varlimsup_{n\to\infty} P\{\|S_{[nt]}^{(n)}\| \geqslant \varepsilon\} \leqslant \mu_t\{g:\ \|g\| \geqslant \varepsilon\}.$$

Hence and from (1) we infer that condition (b) of Lemma 2 is satisfied. Finally, by Lemmas 1 and 8, condition (c) of Lemma 2 also holds, which completes the proof.

Remark 2. It is easy to observe that Theorem 2 ensures the existence of a homogeneous stochastic process W with the continuous sample paths and one-dimensional distributions μ_t, $0 \leqslant t \leqslant 1$. The process W has the independent increments; this follows from the fact that for every system $0 \leqslant t_1 < t_2 < \ldots < t_k \leqslant 1$ there exists an n_0 such that for $n \geqslant n_0$ the increments

$$S_{[nt_2]}^{(n)} (S_{[nt_1]}^{(n)})^{-1}, \ldots, S_{[nt_k]}^{(n)} (S_{[nt_{k-1}]}^{(n)})^{-1}$$

consist of the different independent random variables $X_k^{(n)}$.

In the next remark, which is essentially due to Professor C. Ryll-Nardzewski, we show that Theorem 2 remains valid if G is only metrizable, if we restrict ourselves to tight probability measures.

Remark 3. (i) Let G be a metrizable locally compact group, non-necessarily separable. A measurable mapping of a probability space $(\Omega, \mathfrak{S}, P)$ into $(G, \mathscr{B})$ ($\mathscr{B}$ is the Borel σ-algebra in G) will be called a ***random element*** if its distribution is tight. It is known that if X and Y are random variables defined on a complete probability space, then XY is also a random variable (see [9]).

(ii) Every separable subset of G is contained in an open (hence closed) separable subgroup $G_0 \subseteq G$.

(iii) If μ is a tight measure, then its support is separable, hence, in virtue of (ii), is contained in an open separable subgroup $G_0 \subseteq G$.

(iv) Let M be a separable subset in the space of all tight distributions on G (endowed with the weak topology). Then there exists an open separable subgroup $G_0 \subseteq G$ such that $C(\mu)$ (the support of μ) is contained in G_0 for every $\mu \in M$. In particular, if $(\mu_t)_{t>0}$ is a continuous semigroup of tight probability measures, then $(\mu_t)_{t>0}$ is a separable family; hence, there exists an open separable subgroup $G_0 \subseteq G$ such that $C(\mu_t) \subseteq G_0$ for every $t > 0$.

(v) The space D_G, defined as in the Preliminaries (for non-necessarily separable G), is metric. Every separable subset of D_G is contained in D_{G_0}, where G_0 is an open separable subgroup of G.

(vi) A measurable mapping defined on a probability space $(\Omega, \mathfrak{S}, P)$ into a space D_G with the Borel σ-algebra is called a *random element* (with values in D_G) if its distribution is tight. It is easy to see that if $X_1, X_2, \ldots$ $\ldots, X_n$ is a finite family of random variables defined on a complete probability space, then the mapping

$$\xi_n(t) = X_1 X_2 \ldots X_{[nt]}$$

defines a random element with values in D_G.

(vii) Countable families of random elements with values in D_G are concentrated on D_{G_0}, where G_0 is an open separable subgroup of G. Observe that D_{G_0} is a closed subspace of D_G.

(viii) Using (i)-(vii) we see that Theorem 2 remains valid without assumption of separability of G whenever the probability space, on which considered random variables (elements) are defined, is complete.

COROLLARY 3. *Let μ be a Gaussian measure with a continuous symmetric embedding semigroup $(\mu_t)_{t>0}$. Let $\{\xi(t)\colon t \in [0,1]\}$ be a G-valued homogeneous and separable (in the sense of Doob) stochastic process with independent increments. Assume that $\xi(t)$ has the distribution μ_t for $t \in [0,1]$. Then ξ has the continuous sample paths with probability one whenever $(\Omega, \mathfrak{S}, P)$ is a complete measure space.*

Proof. This follows from Theorem 9.2 in [1] and from the existence of a homogeneous random element W with independent increments and continuous sample paths, which is guaranteed by Theorem 2.

COROLLARY 4. *Let $\{\xi(t)\colon t \in [0,1]\}$ be a homogeneous G-valued stochastic process with independent increments and continuous sample paths. Then ξ is Gaussian.*

Proof. Let μ_t be the distribution of $\xi(t)$. Using Lemma 2 from [2] we obtain

$$\lim_{n\to\infty} \frac{1}{t_n} \mu_{t_n}(G \setminus V) = 0$$

for every open neighbourhood V of the identity of G and for every sequence $t_n \to 0$. Hence μ_1 (and also every μ_t) is a Gaussian measure with an embedding semigroup $(\mu_t)_{t>0}$.

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