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REMARKS ON VECTOR SUMS OF BOREL SETS

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THEOREM. *There is no countably generated (c.g. for brevity) σ-algebra C on the real line which is contained in the Lebesgue measurable sets and has the following property: If B_1 and B_2 are Borel, then their vector sum B_1+B_2 is in C.*

Proof. Let, if possible, C be such a one. Since Borel sets belong to C and every set in C is Lebesgue measurable and C is c.g., it follows that there is a Borel set Z, the complement of which is of measure zero and such that C_Z — which is C restricted to Z — coincides with the collection of Borel sets of Z. No loss in assuming Z contains rationals. Now take an independent perfect set $P \subset Z$ such that the subgroup generated by P is also contained in Z [2]. By repeating the arguments of [1] note that there are two Borel sets B_1, B_2 such that $B_1+B_2 \subset Z$ and $B_1+B_2 \notin C_Z$ and hence $\notin C$.

Remarks. 1. Our theorem — of course motivated by [1] and [3] — complements the theorem obtained in [1] and [3], and which is also due independently to C. A. Rogers.

2. By replacing Lebesgue sets with sets having Baire property we can get a similar theorem. We can also replace the vector sum by vector product.

3. As a particular case of the theorem we have: there is no c.g. σ-algebra C such that $B \subset C \subset L$ and which is closed under vector sums. Here B is the class of Borel sets and L that of measurable sets.

4. Does there exist a c.g. σ-algebra $C \supset B$ which is closed under vector sums? (**P 789**) Perhaps not, in view of the fact that, under the Axiom of Determinateness, the above theorem is still true and, moreover, L is then the class of all subsets of the real line. However, if the word c.g. is omitted, there are many: one can take the class of all subsets of

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reals, but one can get also non-trivial examples of such σ-algebras generated by c-many elements. One has only to start with $\boldsymbol{B}$ and repeat the processes of forming σ-algebras and vector sums till one reaches the first uncountable ordinal.

5. Let X be a complete separable metric space without isolated points. Let $f\colon X\times X\to X$ be a Borel map such that for each fixed x, $f(x,\cdot)$ is a Borel isomorphism of X and so is $f(\cdot, y)$ for each fixed y. One may assume even that they are homeomorphisms. Then does there exist a Borel set $A\subset X$ such that $f(A\times A)$ is not Borel? (**P 790**) If this were true, this would be another generalization of [1], free of group theoretic concepts, besides being a complement to a result of Roger Purves which ensures a Borel set $B\subset X\times X$ such that $f(B)$ is non-Borel.

REFERENCES

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