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APPROXIMATIVE RETRACTS AND FUNDAMENTAL RETRACTS

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In the present note we shall establish a relation between the notion of an approximative retract introduced by H. Noguchi ([5], p. 20) and the Borsuk's notion of a fundamental retract ([1], p. 58, 59 and 65).

By a *space* we understand here always a metric space and by a *map* — a continuous function. A subset A of a space B is said to be an *approximative retract* of B if for every $\varepsilon > 0$ there exists a map $r_\varepsilon\colon B \to A$, called an *$\varepsilon$-retraction*, such that $\varrho\big(r_\varepsilon(x), x\big) < \varepsilon$ for every point $x \in A$. A subset A of a space B is said to be an *approximative neighbourhood retract* of B if A is an approximative retract of some open subset of B. A compactum A is said to be an *approximative absolute retract* (briefly $A \in$ AAR) provided that for every space M and for every homeomorphism $h\colon A \to h(A) \subset M$ the set $h(A)$ is an approximative retract of M. A compactum A is said to be an *approximative absolute neighbourhood retract* (briefly $A \in$ AANR) if for every space M and for every homeomorphism $h\colon A \to h(A) \subset M$ the set $h(A)$ is an approximative neighbourhood retract of M.

Let A and B be two compact subsets of the Hilbert cube Q such that $A \subset B$. The set A is said to be a *fundamental retract* of B ([1], p. 58 and 59) if there exists a sequence of maps $r_n\colon Q \to Q$ satisfying the following conditions:

(i) $r_n(x) = x$ for every point $x \in A$ and for $n = 1, 2, \ldots$

(ii) for every neighborhood V of A in Q there is a neighborhood U of B in Q such that $r_n/U \simeq r_{n+1}/U$ in V for almost all n.

This sequence is called a *fundamental retraction* of the set B onto A and is denoted by $\boldsymbol{r} = \{r_n, B, A\}$.

A compactum A is said to be a *fundamental absolute retract* (briefly $A \in$ FAR) ([1], p. 65) if it is a fundamental retract of Q. A compactum A is said to be a *fundamental absolute neighborhood retract* (briefly $A \in$ FANR) ([1], p. 65) if it is a fundamental retract of some closed neighborhood of A in Q.

The definitions of FAR-sets and FANR-sets differ slightly from those in [1]; however, as follows from [1], the classes of FAR's and FANR's considered here are the same as in [1].

THEOREM. *Every compact approximative retract of a compact subset X of the Hilbert cube Q is a fundamental retract of X.*

Proof. Let us suppose that a compact set A is an approximative retract of the set X. Then for every $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon: X \to A$.

Let $\{P_n\}$ be a sequence of neighborhoods of the set A in Q satisfying the following conditions: $\varrho(x, A) < 1/n$ for every $x \in P_n$ and $n = 1, 2, \ldots$ and $P_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$

We can define P_n to be a prism (in the sense of [2], p. 105). The inclusion $i_n: A \to P_n$, $n = 1, 2, \ldots$, has a continuous extension $f_n: X \to P_n$ ([3], p. 11).

Since $P_n \in \mathrm{ANR}$ and A is a compact set, the function space P_n^A is an ANR-set ([4], p. 260) and therefore it is locally arcwise connected. Thus there exists a positive number ε such that $i_1 r_\varepsilon / A \simeq i_1$ in P_1. Consequently, $f_n i_1 r_\varepsilon / A \simeq f_n i_1$ in P_n. Since $f_n i_1 r_\varepsilon / A = i_n r_\varepsilon / A$ and $f_n i_1 = i_n$, $i_n r_\varepsilon / A \simeq i_n$ in P_n. The map $i_n r_\varepsilon: X \to P_n$ is a continuous extension of the map $i_n r_\varepsilon / A$. As follows by [2], p. 104, also the map i_n has a continuous extension $\hat{r}_n: X_n \to P_n$ such that $\hat{r}_n \simeq i_n r$ in P_n.

Let us notice that $\hat{r}_{n-k} \simeq \hat{r}_n$ in P_{n-k} for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n-1$.

We shall define a sequence of maps $r_n: Q \to Q$ and a sequence $\{U_n\}$ of closed neighborhoods of the set X in the cube Q such that

(1) $\quad r_n(x) = \hat{r}_n(x) \quad$ for every $x \in X$ and $n = 1, 2, \ldots$,

(2) $\quad U_{n+1} \subset U_n \quad$ for $n = 1, 2, \ldots$,

(3) $\quad r_n / U_k \simeq r_k / U_k \quad$ in P_k for $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$

Since $P_1 \in \mathrm{ANR}$, there exists a closed neighborhood U_1 of the set X in Q such that the map $\hat{r}_1: X \to P_1$ has a continuous extension $r_1^0: U_1 \to P_1$. Let $r_1: Q \to Q$ be a map such that $r_1(x) = r_1^0(x)$ for every point $x \in U_1$.

Let us suppose that the sets $U_1, U_2, \ldots, U_{n-1}$ and the maps $r_1, \ldots, r_{n-1}$ satisfying the conditions (1), (2), (3) have been defined. We shall define the set U_n and the map r_n as follows:

Since $P_n \in \mathrm{ANR}$, there exists a closed neighborhood $\hat{U}_n \subset U_{n-1}$ of the set X in Q and a continuous extension $r_n': \hat{U}_n \to P_n$ of the map $\hat{r}_n: X \to P_n$. Let $A_n = \big(U_{n-1} \times (0)\big) \cup \big(\hat{U}_n \times (1)\big) \cup (X \times \langle 0, 1\rangle)$ and let $g_n: A \to P_{n-1}$ be a map defined by the formula:

$$g_n(x, t) = \begin{cases} r_n'(x) & \text{for } t = 1 \text{ and } x \in \hat{U}_n, \\ \varphi_n(x, t) & \text{for } 0 \leqslant t \leqslant 1 \text{ and } x \in X, \\ r_{n-1}(x) & \text{for } t = 0 \text{ and } x \in U_{n-1}. \end{cases}$$

where φ_n is a homotopy between the maps $\hat{r}_n$ and $\hat{r}_{n-1}$ in P_{n-1}. The induction hypothesis (1) with respect to r_{n-1} implies that g_n is continuous. Since $P_{n-1}\in$ ANR, there exists a neighborhood G_n of the set A_n in $Q\times\langle 0,1\rangle$ and a continuous extension $\hat{g}_n\colon G_n\to P_{n-1}$ of the map $g_n\colon A_n\to P_{n-1}$.

Let U_n be a closed neighborhood of X in Q such that $U_n\subset U_{n-1}\cap\hat{U}_n$ and $U_n\times\langle 0,1\rangle\subset G_n$. Set $r_n^0(x)=\hat{g}_n(x,1)$ for every point $x\in U_n$. Since $r_{n-1}(U_{n-1})\subset P_{n-1}$ and $r_{n-1}/U_n\simeq r_n^0$ in P_{n-1}, then applying the homotopy extension theorem ([2], p. 94), we obtain a map $r_n^1\colon U_{n-1}\to P_{n-1}$ such that $r_n^1(x)=r_n^0(x)$ for $x\in U_n$ and $r_n^1\simeq r_{n-1}/U_{n-1}$ in P_{n-1}. Since $U_k\subset U_{k-1}$, $P_k\subset P_{k-1}$, and $P_k\in ANR$, then, applying again the homotopy extension theorem, we can define inductively a sequence of maps $r_n^k\colon U_{n-k}\to P_{n-k}$ for $k=1,2,\ldots,n-1$ such that $r_n^k(x)=r_n^{k-1}(x)$ for $x\in U_{n-1+k}$ and $r_n^k\simeq r_{n-1}/U_{n-k_1}$ in P_{n-k}.

Let $r_n\colon Q\to Q$ be a map such that $r_n(x)=r_n^{n-1}(x)$ for $x\in U_1$. The map r_n satisfies conditions (1) and (3).

If V is a neighborhood of A in Q, then there exists a natural number n_0 such that $P_n\subset V$ for every $n\geqslant n_0$. Then, by condition (3), for every natural number k we have $r_{n_0+k}/U_{n_0}\simeq r_{n_0}/U_{n_0}$ in V. Hence $\boldsymbol{r}=\{r_n,X,A\}$ is a fundamental sequence. If we recall that $r_n(x)=x$ for every point $x\in A$, we infer that $\boldsymbol{r}$ is a fundamental retraction of X to A.

COROLLARY. *If $A\in AAR$ and $A\subset Q$, then $A\in FAR$. If $A\in AANR$ and $A\subset Q$, then $A\in FANR$.*

Note that there exists a compact set $A\in FAR$ which is not an $AANR$. It suffices to set

$$A=\Big\{(x_1,x_2,\ldots)\in Q\colon 0\leqslant x_1\leqslant 1,\ x_2=\frac{1}{8}\Big(\sin\frac{1}{x}+1\Big),$$
$$x_n=0 \text{ for } n=3,4,\ldots\Big\}\cup$$
$$\cup\Big\{(x_1,x_2,\ldots)\in Q\colon 0\leqslant x_2\leqslant\frac{1}{2},\ x_n=0 \text{ for } n\neq 2\Big\}.$$

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