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FREE INVERSE SEMIGROUPS

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1. Introduction and summary. The existence of a free inverse semigroup on a set X was first established by Vagner [8] by an argument reminiscent of the construction of a free group on X. A concrete construction was first offered by Scheiblich [6]. This was followed by various other constructions due to Munn [3], Preston [4] and Schein [7]. A characterization of a free semilattice of groups was announced by Liber [1].

It was noticed by McAlister and McFadden [2] that a free inverse semigroup can be represented as a P-semigroup introduced by McAlister. It has also been observed by several authors that a free inverse semigroup on X with an identity adjoined is a free inverse monoid on X, so only the latter may be considered.

We provide here a simple direct proof of the fact that a free inverse monoid on X can be represented in a form similar to that of a P-semigroup by exhibiting a homomorphism of $(X \cup X')^*$ onto this semigroup constructed by means of a semilattice and a free group on X. Even though the end result is not new, our proof makes the interplay of the words on X and the pairs in this representation explicit, rendering the representation more transparent. Hence we deduce the construction of a free semilattice (which is well known), and the free semilattice of groups due to Liber. As a further application, we furnish the Munn representation of the elements of a free inverse semigroup by word trees with a multiplication in terms of graphs making this representation an isomorphism. This solves a problem posed by Reilly [5].

2. The construction. Let X be a fixed nonempty set. A *free inverse semigroup on* X is a pair (F_X, φ), where F_X is an inverse semigroup, φ is an injection of X into F_X such that, for every inverse semigroup S and a function ψ of X into S, there exists a unique homomorphism $\bar{\psi}$ from F_X into S making the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & F_X \arrow[dl, "\bar{\psi}"] \\ & S & \end{tikzcd}$$

commutative.

As we mentioned above, for the purposes of our discussion, we may restrict our attention to monoids. Note that a monoid is a semigroup with an identity element and that a homomorphism of monoids S and T maps the identity of S onto the identity of T. In an obvious way, we may speak of a *free inverse monoid on* X.

Now let φ be a bijection of the set X onto a set X' disjoint from X. Let $Z = (X \cup X')^*$ be the set of all words on the set $X \cup X'$, including the empty word, here denoted by 1, under juxtaposition as multiplication (the free monoid on $X \cup X'$). For any $x \in X \cup X'$ let

$$x^{-1} = \begin{cases} x\varphi & \text{if } x \in X, \\ x\varphi^{-1} & \text{if } x \in X^{-1}, \end{cases} \qquad 1^{-1} = 1,$$

and for $x_1 x_2 \ldots x_n \in Z$

$$(x_1 x_2 \ldots x_n)^{-1} = x_n^{-1} \ldots x_2^{-1} x_1^{-1}.$$

It is clear that $w \to w^{-1}$ is an involution (antiautomorphism whose square is the identity) of Z, and it was noted in [7] that $(Z, {}^{-1})$ is a free involuted monoid on X. Note that a homomorphism of involuted semigroups must preserve the involution. Here and above, we call the monoid alone a free object since the injection from X into Z is obvious. We will do this also for some other free monoids.

From now on, we write Z for the monoid with involution just constructed. Let ϱ be the congruence on Z generated by the relation

$$\{(uu^{-1}u, u) \mid u \in Z\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in Z\}.$$

Then $(Z/\varrho, \varrho^{\#}|_X)$ is a free inverse monoid on X; this is the original description of Vagner [8] (for a proof see [5]). Call ϱ the *Vagner congruence* on Z.

For any $w \in Z$, let $r(w)$ denote the *reduced word* of w, which is obtained by successive omission of all occurrences of subwords of w of the form xx^{-1} with $x \in X \cup X'$. The set G_X of all reduced words with the multiplication $w \cdot u = r(wu)$ is a *free group* on X. Again the identity is the empty word, which will be denoted by 1.

For any $w = x_1 x_2 \ldots x_n \in Z$ let

$$\hat{w} = \{1, x_1, x_1 x_2, \ldots, x_1 x_2 \ldots x_n\}.$$

A subset A of G_X is *closed* if $w \in A$ implies $\hat{w} \subseteq A$. Let E denote the set of all finite closed subsets of G_X. Finally, put

$$I_X = \{(A, g) \in E \times G_X \mid g \in A\}$$

with the multiplication $(A, g)(B, h) = (A \cup gB, gh)$, where both gB and gh are products computed in G_X.

It will follow from the theorem below that I_X is a monoid. One verifies easily that $(g^{-1}A, g^{-1})$ is an inverse of (A, g), and that idempotents of I_X commute. Hence I_X is an inverse semigroup.

The elements of $\hat{w}$ are the *left factors* of w. The reduced left factors of w form the set

$$\mathrm{rlf}(w) = \{1, r(x_1), r(x_1x_2), \ldots, r(x_1x_2\ldots x_n)\}.$$

THEOREM 1. *The function*

$$\psi\colon w \to (\mathrm{rlf}(w), r(w)) \qquad (w \in Z)$$

is an (involution preserving) homomorphism of Z onto I_X which induces the Vagner congruence on Z.

Proof. In order to verify that ψ maps Z into I_X, it suffices to show that, for $w = x_1x_2\ldots x_n \in Z$, $\mathrm{rlf}(w)$ is closed. Hence let $u \in \mathrm{rlf}(w)$. Then $u = r(x_1x_2\ldots x_i)$ for some $1 \leqslant i \leqslant n$, so $u = x_{j_1}x_{j_2}\ldots x_{j_m}$ for some $1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant i$. For any $1 \leqslant k \leqslant m$ we have

$$x_{j_1}x_{j_2}\ldots x_{j_k} = r(x_1x_2\ldots x_{j_k}) \in \mathrm{rlf}(w).$$

Hence $\mathrm{rlf}(w)$ is closed.

For $w = x_1x_2\ldots x_m$ and $u = y_1y_2\ldots y_n$ we obtain

$$\begin{aligned}(w\psi)(u\psi) &= (\mathrm{rlf}(w), r(w))(\mathrm{rlf}(u), r(u))\\ &= (\mathrm{rlf}(w) \cup r(w)\,\mathrm{rlf}(u), r(w)\cdot r(u))\\ &= (\{1, x_1, r(x_1x_2), \ldots, r(w)\} \cup \{r(w), r(wy_1), \ldots, r(wu)\}, r(wu))\\ &= (\mathrm{rlf}(wu), r(wu)) = (wu)\psi.\end{aligned}$$

In order to see that $(w\psi)^{-1} = w^{-1}\psi$ it suffices to show that

$$r(w)^{-1}\,\mathrm{rlf}(w) = \mathrm{rlf}(w^{-1}).$$

Indeed,

$$\begin{aligned}r(w)^{-1}\,\mathrm{rlf}(w) &= r(x_n^{-1}x_{n-1}^{-1}\ldots x_1^{-1})\{1, x_1, r(x_1x_2), \ldots, r(w)\}\\ &= \{r(x_n^{-1}\ldots x_1^{-1}), r(x_n^{-1}\ldots x_1^{-1})x_1, \ldots, r(x_n^{-1}\ldots x_1^{-1})r(w)\}\\ &= \{r(w^{-1}), \ldots, r(x_n^{-1}x_{n-1}^{-1}), x_n^{-1}, 1\} = \mathrm{rlf}(w^{-1}).\end{aligned}$$

Hence ψ is a homomorphism of involuted semigroups.

Let $(A, g) \in I_X$. Then $A = \{w_1, w_2, \ldots, w_n\}$ for some $w_i \in G_X$, and since $g \in A$, we may take $g = w_n$. We now let

$$w = (w_1w_1^{-1})(w_2w_2^{-1})\ldots(w_{n-1}w_{n-1}^{-1})w_n.$$

It is clear that $A = \mathrm{rlf}(w)$ and that $r(w) = w_n = g$ so that $(A, g) = w\psi$. Thus ψ maps Z onto I_X.

As a consequence, I_X is a monoid.

Let τ be the congruence on Z induced by ψ and let ϱ be the Vagner congruence on Z. Since I_X is an inverse monoid, we have $\varrho \subseteq \tau$ by the minimality of ϱ. For the opposite inclusion, we proceed as follows.

For $w = x_1 x_2 \ldots x_n \in Z$, letting

$$\bar{w} = (x_1 x_1^{-1})[r(x_1 x_2) r(x_1 x_2)^{-1}] \ldots [r(x_1 \ldots x_{n-1}) r(x_1 \ldots x_{n-1})^{-1}] r(w),$$

we claim that $w \varrho \bar{w}$. The argument is by induction on n. This is trivial for $n = 0, 1$. Let $n > 1$ and assume the statement is true for all $k < n$. Then

$$\begin{aligned} x_1 x_2 \ldots x_n \varrho (x_1 x_1^{-1}) \ldots [r(x_1 \ldots x_{n-2}) r(x_1 \ldots x_{n-2})^{-1}] r(x_1 \ldots x_{n-1}) x_n \\ \varrho (x_1 x_1^{-1}) \ldots [r(x_1 \ldots x_{n-1}) r(x_1 \ldots x_{n-1})^{-1}] [r(x_1 \ldots x_{n-1}) x_n] \\ = (x_1 x_1^{-1}) \ldots [r(x_1 \ldots x_{n-1}) r(x_1 \ldots x_{n-1})^{-1}] r(x_1 \ldots x_n), \end{aligned}$$

which proves that $w \varrho \bar{w}$. Now assume that $w \tau u$. Then $w\psi = u\psi$ so that $\mathrm{rlf}(w) = \mathrm{rlf}(u)$ and $r(w) = r(u)$. It follows easily that $\bar{w} \varrho \bar{u}$, which by the above yields $w \varrho u$. Hence $\tau = \varrho$ and ψ indeed induces ϱ.

As a consequence of this theorem and the remarks made just before it, we have

COROLLARY. *The pair (I_X, φ), where φ: $x \to (\hat{x}, x)$, is a free inverse monoid on X.*

3. Free semilattices. It is well known that a free semilattice on a set X admits a representation as the set of all finite nonempty subsets of X under the operation of set-theoretical union. If we also include the empty set, we obtain a free monoid semilattice on X, which will be denoted by Y_X. Note that the identity mapping on Y_X is an involution.

We may obtain this representation from Z discussed above by first obtaining an analogue of the Vagner congruence for this case. In fact, let η be the congruence on Z generated by the relation

$$\{(u^2, u) \mid u \in Z\} \cup \{(uv, vu) \mid u, v \in Z\}.$$

Then a simple argument shows that $(Z/\eta, \eta^{*}|_X)$ is a free monoid semilattice on X.

In order to get a homomorphism of Z onto Y_X which induces η we introduce the following concept. For each $w = x_1 x_2 \ldots x_n \in Z$ define the *content* of w by

$$c(w) = \{|x_1|, |x_2|, \ldots, |x_n|\},$$

where

$$|x| = \begin{cases} x & \text{if } x \in X, \\ x^{-1} & \text{if } x \in X', \end{cases} \qquad |1| = 1.$$

PROPOSITION 1. *The mapping c: $w \to c(w)$ is a homomorphism of Z onto Y_X which induces η on Z.*

Proof. The mapping c is clearly multiplicative. For any $x \in X$ and $y \in X'$ we have

$$c(x^{-1}) = \{x\} = c(x) = c(x)^{-1},$$
$$c(y^{-1}) = \{y^{-1}\} = c(y) = c(y)^{-1},$$

which implies $c(w^{-1}) = c(w)^{-1}$ for all $w \in Z$. It follows that c is a homomorphism of Z onto Y_X. Letting τ be the congruence on Z induced by c, we get $\eta \subseteq \tau$ by the minimality of η. The opposite inclusion follows without difficulty.

As a corollary we infer that Y_X is a free monoid semilattice on X. In the notation introduced in the preceding section, we may extend the concept of content to any subset A of E by letting

$$c(A) = \bigcup_{w \in A} c(w).$$

As another consequence, we obtain

COROLLARY. *The mapping $(A, g) \to c(A)$ is a homomorphism of I_X onto Y_X which induces the least semilattice congruence on I_X.*

Proof. It suffices to observe that for any $w \in Z$ we have $c(w) = c(\mathrm{rlf}(w))$.

4. Free semilattice of groups. We call semilattices of groups *completely inverse semigroups* (they are precisely *completely regular inverse semigroups*). We can proceed here as in the preceding sections. Indeed, let ν be the congruence on Z generated by the relation

$$\{(uu^{-1}u, u) \mid u \in Z\} \cup \{(uvv^{-1}, vv^{-1}u) \mid u, v \in Z\}.$$

Then an argument analogous to the one for inverse semigroups yields that $(Z/\nu, \nu^{\#}|_X)$ is a *free completely inverse monoid* on X.

PROPOSITION 2. *The mapping φ: $w \to (c(w), r(w))$ $(w \in Z)$ is a homomorphism of Z onto the subsemigroup*

$$C_X = \{(A, u) \in Y_X \times G_X \mid c(u) \subseteq A\}$$

of $Y_X \times G_X$ which induces ν on Z.

Proof. It is easy to see that φ is a homomorphism of Z into C_X. If $(A, u) \in C_X$, then

$$u = x_1 x_2 \ldots x_m \quad \text{and} \quad A = \{x_1, x_2, \ldots, x_m, \ldots, x_n\},$$

so

$$\big(u(x_{m+1}x_{m+1}^{-1}) \ldots (x_n x_n^{-1})\big)\varphi = (A, u).$$

Hence φ maps Z onto C_X.

It follows that C_X is closed under multiplication and inversion, so is an inverse semigroup. Being a subdirect product of a semilattice and a group, it must be a completely inverse semigroup.

Now let τ be the congruence on Z induced by φ. Then by the minimality of ν we have $\nu \subseteq \tau$. For $w = x_1 x_2 \dots x_n$ we let

$$w^* = (|x_1|\,|x_1|^{-1}) \dots (|x_n|\,|x_n|^{-1})\, r(w)$$

and observe that $w \,\nu\, w^*$. If now $w \,\tau\, u$, then $c(w) = c(u)$ and $r(w) = r(u)$, which implies $w^* \,\nu\, u^*$, and thus $w \,\tau\, u$. Consequently, $\tau \subseteq \nu$, and the equality holds true.

COROLLARY 1. *The pair* (C_X, θ), *where* θ: $x \to (\{x\}, x)$, *is a free completely inverse monoid on* X. *Furthermore,* C_X *is a subdirect product of* Y_X *and* G_X.

COROLLARY 2. *The mapping* $(A, g) \to (c(A), g)$ *is a homomorphism of* I_X *onto* C_X *which induces the least completely inverse congruence on* I_X. *Moreover, the intersection of the least semilattice congruence and the least group congruence equals the least completely inverse congruence both in* Z *and in* I_X.

We can now derive the form of Liber [1] of a free completely inverse monoid on X as follows. The notation $[Y;\ G_\alpha,\ \varphi_{\alpha,\beta}]$ stands for a completely inverse semigroup S which is a semilattice Y of groups G_α with connecting homomorphisms $\varphi_{\alpha,\beta}$. As above, let G_A denote the free group on A, and for $A \subseteq B$ let $i_{A,B}$: $G_A \to G_B$ be the inclusion mapping. We readily deduce that

$$C_X \cong [Y;\ G_A,\ i_{A,B}].$$

5. Graph representation. Here we follow the notation and terminology of Munn [3] and refer the reader to this paper for a complete discussion of this subject. We will list now only a few most essential concepts we will need. The only deviation from [3] is the fact that we consider here a free inverse monoid over X instead of a free inverse semigroup over X.

A *tree* is a connected graph without cycles. A *word tree* T on X is a finite tree with oriented edges labelled by elements of X and having no subgraph of the form

$$\circ \xrightarrow[x]{} \circ \xleftarrow[x]{} \circ \quad \text{or} \quad \circ \xleftarrow[x]{} \circ \xrightarrow[x]{} \circ$$

The set of labels is extended to $X \cup X'$ by the convention that

$$\circ \xrightarrow[x^{-1}]{} \circ$$

has the same meaning as

$$\circ \xleftarrow[x]{} \circ\ .$$

Let T be a word tree on X. We denote by $V(T)$ the set of vertices of T. Let $\alpha, \beta \in V(T)$. An (α, β)-*path* on T is a sequence

$$(\alpha = \gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n = \beta)$$

of distinct vertices of T such that γ_{i-1}, γ_i are adjacent for $i = 1, 2, \dots, n$; we denote it by $\Pi(\alpha, \beta)$.

An *isomorphism* of word trees T and T' is a bijection of $V(T)$ and $V(T')$ which preserves adjacency, orientation of edges, and labelling of edges. Obviously, isomorphism is an equivalence relation on the class of all word trees. Let $\mathscr{T}_X$ be a transversal of the isomorphism classes of word trees on X.

Let $T, T' \in \mathscr{T}_X$ and $\alpha \in V(T)$, $\alpha' \in V(T')$. Let $\gamma \in V(T')$ and let

$$(\alpha' = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma)$$

be the (α', γ)-path in T'. There exists $\delta \in V(T)$ such that

$$\Pi(\alpha, \beta) = (\alpha = \delta_0, \delta_1, \ldots, \delta_m = \delta)$$

is isomorphic to $(\alpha' = \gamma_0, \gamma_1, \ldots, \gamma_m)$ and m is the greatest integer with this property. We now identify $\gamma_i = \delta_i$ for $i = 0, 1, 2, \ldots, m$. If $m < n$, we attach the graph $(\gamma_m, \gamma_{m+1}, \ldots, \gamma_n)$ to the vertex $\gamma_m = \delta_m$. Doing this for every $\gamma \in V(T')$, we evidently obtain a word tree on X. Let $T(\alpha, \alpha') T'$ denote its representative in $\mathscr{T}_X$. It is a notational convenience to identify the vertices of T and T' with the corresponding vertices of $T(\alpha, \alpha') T'$.

A triple (T, α, β) is a *birooted word tree* on X if $T \in \mathscr{T}_X$ and $\alpha, \beta \in V(T)$. Let $\mathscr{B}\mathscr{T}_X$ denote the set of all birooted word trees on X together with the multiplication

$$(T, \alpha, \beta)(T', \alpha', \beta') = \big(T(\beta, \alpha') T', \alpha, \beta'\big).$$

The convention at the close of the preceding paragraph guarantees that $\mathscr{B}\mathscr{T}_X$ is closed under this operation. It is remarkable that this multiplication has a very similar form to that in a Rees matrix semigroup $\mathscr{M}(G;\ I, \Lambda;\ P)$! We construct next a function from I_X into $\mathscr{B}\mathscr{T}_X$ which will turn out later to be an isomorphism.

First note that the length of a word $w = x_1 x_2 \ldots x_n \in Z$ is equal to n. Hence the length of 1 is zero. We let $(A, g) \in I_X$. We will construct inductively a birooted word tree on X associated to (A, g).

To 1 we associate a vertex α. Let $w \in A$ be of length 1. Form an edge (α, γ) labelled by x. Keep α fixed and apply the same procedure to all other words in A of length 1, thus obtaining the edges of the form $(\alpha, \delta), \ldots$ Assume that we have assigned a path, starting at α, to each word in A of length less than k, and let $w = x_1 x_2 \ldots x_k \in A$. Observe that there exists a unique path $(\alpha = \gamma_0, \gamma_1, \ldots, \gamma_{k-1})$ labelled by $x_1, x_2, \ldots, x_{k-1}$ in the graph already constructed. Attach an edge (γ_{k-1}, γ_k) labelled by x_k. Do this for all words in A of length k.

We are thus able to construct inductively a word tree. Denote by T its representative in $\mathscr{T}_X$ and let β be the vertex of T for which the (α, β)-path is the one labelled by $x_1, x_2, \ldots, x_n$, where $g = x_1 x_2 \ldots x_n$. In this way, we arrive at a birooted word tree (T, α, β) on X and writte $(A, g)\tau = (T, \alpha, \beta)$.

THEOREM 2. *The function τ just defined is an isomorphism of I_X onto $\mathscr{B}\mathscr{T}_X$.*

Proof. First note that, for $(A, g) \in I_X$, $(A, g)\tau$ is the unique birooted word tree (T, α, β) on X for which the set of all (α, γ)-paths bears the labels of words in A (the (α, α)-path corresponds to 1) and (α, β)-path bears the label of g. Hence τ is a function mapping I_X into $\mathscr{B}\mathscr{T}_X$.

Next let $(T, \alpha, \beta) \in \mathscr{B}\mathscr{T}_X$. Let A be the set of all words in Z which label the (α, γ)-paths as γ runs over $V(T)$, and let g be the word which labels $\Pi(\alpha, \beta)$. Note that any edge with a "wrong" orientation labelled x can be replaced by an edge with a "right" orientation labelled x^{-1}. Clearly, with every $w \in A$, all the left factors of w are also in A. Since also $g \in A$, we get $(A, g) \in I_X$. It is easy to see that $(A, g)\tau = (T, \alpha, \beta)$ and (A, g) is the only element of I_X with this property. Consequently, τ is a bijection of I_X onto $\mathscr{B}\mathscr{T}_X$.

For (A, g), $(A', g') \in I_X$ we have

$$(A, g)(A', g') = (A \cup gA', gg');$$

let $(A, g)\tau = (T, \alpha, \beta)$ and $(A', g')\tau = (T', \alpha', \beta')$. The word tree corresponding to $A \cup gA'$ is obtained by taking the union of T and T' shifted by $\Pi(\alpha, \beta)$, that is, the union of T and T' with α' being identified with β. But this is the word tree $T(\beta, \alpha')T'$ constructed before the present theorem. Since β and α' have been identified, the path $\Pi(\alpha, \beta')$ can be obtained by following $\Pi(\alpha, \beta)$ by $\Pi(\alpha', \beta')$. This shows that gg' labels $\Pi(\alpha, \beta')$. Therefore τ is also a homomorphism.

We may combine Theorems 1 and 2 to deduce easily some of the principal results in [3]; we omit the details. For a word tree T, let

$$l(T) = \{|x| \mid x \text{ labels an edge of } T\}.$$

From the discussion in Sections 3 and 4 we then get, for $U = (T, \alpha, \beta)$ and $U' = (T', \alpha', \beta')$,

$$U \sigma U' \Leftrightarrow \Pi(\alpha, \beta) \cong \Pi(\alpha', \beta'), \qquad U \eta U' \Leftrightarrow l(U) = l(U'),$$
$$U \nu U' \Leftrightarrow \Pi(\alpha, \beta) \cong \Pi(\alpha', \beta'), \qquad l(U) = l(U'),$$

where σ and η are the least group and semilattice congruences on $\mathscr{B}\mathscr{T}_X$, respectively, and $\nu = \sigma \cap \eta$ is the least completely inverse congruence on $\mathscr{B}\mathscr{T}_X$.

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