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A NON-DEGENERATE σ-DISCRETE MOORE SPACE WHICH IS CONNECTED

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A space is *σ-discrete* if it is the union of a countable number of sets none of which has a limit point in the space. It is well known that no non-degenerate connected Moore space can have only countably many points and it might seem a natural generalization that no such space could be σ-discrete. This note* provides an example to show that such a generalization is not valid.

First we define a Moore space S' which is σ-discrete but not connected. The construction follows closely that of Bing [1]. We then enlarge S' to the desired space S.

Let $V_1, V_2, \ldots$ be distinct vertical lines in the plane whose union is dense in the plane. The points of S' are the points in this union. For each point p in S' and for each positive integer n, the *region of S' centred at p of size n*, denoted by $R_n(p)$, is the set containing only p and those points of S' which lie in the interior of the largest circle in the plane which

(1) is tangent at p to the vertical line in the plane containing p,

(2) lies, except for p, to the right of this vertical line,

(3) has diameter less than or equal to one and contains in its interior no point in any one of $V_1, \ldots, V_n$.

There is a development $G_1', G_2', \ldots$ for S' such that, for each positive integer n, G_n' is the set of all regions of S' which are centred at a point of S' and which have integral size greater than or equal to n.

For each point p of S' define $A(p)$ to be a point set conumerous with the real numbers and such that, if p and q are two points of S', then $A(p)$ intersects neither $A(q)$ nor S'. When convenient, $A(p)$ is referred to as

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(1) R. H. Bing, *Metrization of topological spaces*, Canadian Journal of Mathematics 3 (1951), p. 175-186.

the set of points above p. The union of all the point sets just defined is denoted by S''. The points of S are the points in the union of S', S'' and $\{\omega\}$, where ω is some point which is neither in S' nor in S'', and the topology of S is determined by the development $G_1, G_2, \ldots$ described in the sequel.

For each positive integer i let C_i be a set of circles in the plane such that

(1) each circle in C_i lies to the right of V_i and does not intersect V_i,

(2) no two circles in C_i intersect or have interiors which intersect,

(3) if R' is a region in S' centred at a point in V_i, then there is a circle in C_i the common part of whose interior with S' lies entirely in R'.

Note that C_i is countable. Further, for each positive integer n define C_i^n to be the set to which X belongs only if, for some circle X' in C_i with radius r, X is the circle concentric with X' of radius r/n. If $X_1, X_2, \ldots$ is a sequence in C_i converging in the plane to a point p in V_i, then we denote by $X_1^n, X_2^n, \ldots$ a definite sequence in C_i^n such that

(1) there is a positive integer j such that, for each integer k greater than j, X_{k-j}^n is concentric with X_k,

(2) each of $X_1^n, X_2^n, \ldots$ lies in the interior of a circle in the plane which is centred at p, has radius less than $1/n$ and does not intersect any one of the verticals $V_1, \ldots, V_n$ different from V_i.

In addition, let $\tilde{X}^n$ denote the set to which a point belongs only if it is in S' and it or its reflexion in a horizontal line through p lies in the interior of one of $X_1^n, X_2^n, \ldots$ For each p in V_i let $M(p)$ be a set of sequences in C_i such that

(1) each sequence in $M(p)$ converges in the plane to p,

(2) no term of any sequence in $M(p)$ intersects $R_1(p)$ or contains a point with the second coordinate greater than the second coordinate of p,

(3) no two sequences in $M(p)$ share more than a finite number of terms,

(4) if $X_1, X_2, \ldots$ is any sequence in C_i converging in the plane to p no term of which intersects $R_1(p)$ or has a point with the second coordinate greater than the second coordinate of p, then there is a sequence in $M(p)$ which has infinitely many terms in common with $X_1, X_2, \ldots$

(5) $M(p)$ is conumerous with $A(p)$.

Let W be a one-to-one function with domain S'' such that, for each p in S', $W(A(p))$ is $M(p)$. Finally, G_n is defined to be the set to which R belongs if and only if one of the following holds:

(1) R is the set of all points x in S such that there is a region R' in G_n' centred at p, and x is either in R' or above some point in R' different from p;

(2) R is the set of all points x in S such that there are a point q in S'' and a positive integer m greater than or equal to n, and x is q or x is in $(W(q)^m)^{\sim}$, or x is above some point in $(W(q)^m)^{\sim}$;

(3) R is the set of all points x in S such that x is ω or x is in S' and the second coordinate of x is greater than n, or x is above some point in S' whose second coordinate is greater than n.

For each ordered pair of positive integers (m, n) let $S(m, n)$ denote the set of all x in S such that either x is ω or x is in V_m and has the second coordinate less than n, or x is above some point in V_m whose second coordinate is less than n. No point of S is a limit point of $S(m, n)$ and each point of S is in some $S(m, n)$. It follows that S is σ-discrete.

To see that S is a connected space we suppose that the points of S lie in two mutually separated point sets H and K with ω in H and we show that this leads to a contradiction. Since each point of S which is not in S' is a limit point of S', there is a region R' of S' which lies in K and, consequently, there is a vertical straight line V_i, a segment of which lies entirely in K. Since ω is in H, there is a point p in V_i such that

(1) p is a limit point in the plane of those points of K in V_i which have the second coordinate smaller than the second coordinate of p,

(2) p is not a limit point in the plane of those points of K in V_i which have the second coordinate greater than the second coordinate of p.

Thus we may choose a sequence of points $t_1, t_2, \ldots$ common to K and V_i which converges in the plane to p and such that each of its terms has the second coordinate less than the second coordinate of p. We may choose also a sequence of points $s_1, s_2, \ldots$ common to H and V_i such that, for each positive integer n, s_n is the reflexion of t_n in the horizontal line through p. Let $R'(s_1), R'(s_2), \ldots$ be regions of S' centred at $s_1, s_2, \ldots$, respectively, and lying in $H-R_1(p)$. Let $R'(t_1), R'(t_2), \ldots$ be regions of S' centred at $t_1, t_2, \ldots$, respectively, and lying in $K-R_1(p)$. By construction of C_i there is a sequence $X_1, X_2, \ldots$ in C_i, converging in the plane to p, such that for each positive integer n the common part of S' with the interior of X_n is a subset of $R'(t_n)$, and the common part of S' with the interior of the reflexion of X_n in the horizontal line through p is a subset of $R'(s_n)$. By construction of $M(p)$ there is a sequence $Y_1, Y_2, \ldots$ in $M(p)$ which shares infinitely many terms with the sequence $X_1, X_2, \ldots$ Let q be the point above p such that $W(q)$ is $Y_1, Y_2, \ldots$ If m is a positive integer, then $\left(W(q)^m\right)^{\sim}$ intersects both the union of $R'(s_1), R'(s_2), \ldots$, which is a subset of H, and the union of $R'(t_1), R'(t_2), \ldots$, which is a subset of K. It follows that q is a limit point in S both of H and K. Consequently, H and K are not mutually separated. This is a contradiction.

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