COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 1

RESIDUAL SETS NOT OF MAXIMUM DIMENSION

BY

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1. Preliminaries. Let M be a compact connected topological n-manifold. A subset R of M is said to be *residual* in M if either (a) bdry $M = \emptyset$ and $M-R$ is a topological open n-cell which is dense in M or (b) bdry $M \neq \emptyset$ and $M-R$ is a topological copy of bdry $M \times (0, 1]$ which is dense in M. In [5], Doyle and Hocking showed that residual sets always exist in case (a); if the collaring theorem of Brown [3] is used as a starting point, an analogous argument shows their existence in case (b). The two cases can be treated simultaneously as follows: if bdry $M = \emptyset$, remove a small open n-cell, whose boundary is bicollared in M, from $M-R$ to obtain a new manifold M_B with non-empty boundary. Then R is residual in M_B.

Notation. If bdry $M \neq \emptyset$, define $M_B = M$; if bdry $M = \emptyset$, define M_B as just described.

Using this notation and the terminology of [7] and [8], a set R residual in M will be called a (*topological*) *spine* of M if M_B is homeomorphic with the mapping cylinder of some surjection g: bdry $M_B \to R$. (In this case M_B and R have the same homotopy type, and R is a compact ANR.) Brown and Casler [4] have proved that spines always exist, and, in [8], the proof of the Brown-Casler theorem was modified to show that any residual ANR can be enlarged by an arbitrarily small amount to yield a spine.

Here, it will be shown that as far as cohomology is concerned, a residual ANR is equivalent to a spine; a relationship between the dimension of such a set and the cohomology of the containing manifold is then studied and the results applied to deduce some geometric propositions resembling the Jordan separation theorem.

H^* will denote singular cohomology, $\bar{H}^*$ Alexander-Čech cohomology, and G — a coefficient module over a PID.

2. Residual ANR's and spines. Let R be residual in M, and suppose

$$h\colon \text{bdry } M_B \times (0, 1] \to M_B - R$$

is a homeomorphism onto. Define

$$N_\delta = R \cup h(\text{bdry } M_B \times (0, \delta)) \quad \text{for } 0 < \delta \leqslant 1.$$

If the collection of all open neighborhoods of R is directed by reverse inclusion, then $\{N_\delta\}$ is cofinal in this collection; moreover, if $a < \delta$, the inclusion $N_a \subset N_\delta$ induces an isomorphism of $H^*(N_\delta; G)$ with $H^*(N_a; G)$. Thus, for each δ, $H^*(N_\delta; G) \approx H^*(N_1; G) = H^*(M_B; G)$. Therefore (by proposition 27.1 of [6], for example), we can infer

LEMMA 1. *If R is an ANR residual in M, then*

$$H^*(R; G) \approx \bar{H}^*(R; G) = \varinjlim H^*(N_\delta; G) \approx H^*(M_B; G).$$

That this lemma has wider applicability than merely for the well-known case wherein R is a spine is demonstrated by the following

Example. Let R be the simple arc imbedded in $M = S^3$ as described by Artin and Fox in Example 1.2 of [1]. As shown therein, $M - R$ is homeomorphic with E^3, but the imbedding of R in M is wild. According to Bing and Kirkor [2], if R were to have a mapping cylinder neighborhood, R would be tame in M. Therefore, R is not a spine, for if it were, M_B would be a mapping cylinder neighborhood of R in M. (This answers a question raised in [8]; the results of [10] make possible the construction of many other such examples in dimension 3.)

3. Cohomological dimension. A general reference for this section is the text by Nagami [9], especially p. 199-212 in the appendix by Kodama.

The *large cohomological dimension* $D(X: G)$ of a finite-dimensional Hausdorff space X with respect to an Abelian group G is defined to be the largest integer m such that $\bar{H}^m(X, A; G) \neq 0$ for some closed subset A of X. It is known that $\dim X = D(X: Z)$ if X is compact. I am grateful to C. R. F. Maunder, who pointed out to me the following lemma:

LEMMA 2. *If X is compact, $D(X: Z_p) \leqslant D(X: Z)$.*

Proof. Let $m > D(X: Z)$. By the universal coefficient theorem,

$$\bar{H}^m(X, A; Z_p) \approx \text{Hom}(\bar{H}_m(X, A; Z), Z_p) \oplus \text{Ext}(\bar{H}_{m-1}(X, A; Z), Z_p).$$

This expression is 0 because $\bar{H}_m(X, A; Z) = 0$ and $\bar{H}_{m-1}(X, A; Z)$ is free.

COROLLARY. *Let R be an ANR residual in M. Then*

$$D(M_B: Z_p) \leqslant D(M_B: Z) = \dim R$$

and

$$\bar{H}^m(M; Z) = \bar{H}^m(M; Z_p) = 0 \quad (\dim R < m < n).$$

Proof is immediate from Lemmas 1 and 2.

4. An application to Jordan separation. In this section, M denotes a compact connected n-manifold without boundary, K a compact connected $(n-1)$-manifold without boundary imbedded in M, and R — an ANR residual in M.

THEOREM 1. *If* $\dim R < n-1$, *then K separates M.*

Proof. Since K and M are compact metric ANR's, their respective absolute singular cohomology groups coincide with those obtained from the Alexander-Čech theory. By Alexander duality, $H_0(M-K) \approx \bar{H}^n(M, K)$ (coefficients in Z_2). Using these facts and the results of the preceding sections, the exact cohomology sequence

$$\bar{H}^{n-1}(M) \to \bar{H}^{n-1}(K) \to \bar{H}^n(M, K) \to \bar{H}^n(M) \to \bar{H}^n(K)$$

of the pair (M, K) becomes

$$0 \to Z_2 \to H_0(M-K) \to Z_2 \to 0.$$

This implies that the rank of the Z_2-module $H_0(M-K)$ is at least 2, so $M-K$ is not connected.

In like spirit:

THEOREM 2. *If $\bar{H}^{n-1}(R;\, Z)$ has rank* 0 *and M and K are orientable, then K separates M.*

Proof. Similar to that of Theorem 1, this time using integer coefficients.

Note. Several years ago, P. H. Doyle showed me a purely point--set-theoretic proof of the following:

If M has a residual set Q (not necessarily an ANR) with $\dim Q < n-1$, *then every $(n-1)$-sphere imbedded in M separates M.*

Some tantalizing questions naturally arise:

1. Can Theorem 1 be further generalized to include the case where R is not required to be an ANR? (**P 819**)

2. If R denotes a set residual in M and S a spine of M, is it true that $\min \dim R = \min \dim S$? Can the analogous questions for residual ANR's, or with cohomological dimension replacing dimension, be answered? (**P 820**)

3. If a closed n-manifold M contains a non-separating $(n-1)$--sphere, must every spine of M contain an $(n-1)$-sphere? (The same question with "residual set" in place of "spine" is easily answered in the negative.) (**P 821**)

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Reçu par la Rédaction le 20. 1. 1972