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COMPACTA WHICH ARE QUASI-HOMEOMORPHIC WITH A DISK

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1. Introduction. By a *disk* we mean any set which is homeomorphic with the ball $K = \{x \in E^2\colon |x| \leqslant 1\}$. We shall consider metrizable spaces only. By a *compactum* we mean any compact metric space. The AR-spaces will be assumed to be compact. A map f of a compactum X into a space Y is said to be an *ε-mapping* if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. A compactum X is said to be *quasi-embeddable* into a space Y if for every $\varepsilon > 0$ there is an ε-mapping of X into Y. Given two compacta X and Y, X is said to be *Y-like* if for every $\varepsilon > 0$ there is an ε-mapping of X onto Y. The spaces X and Y are said to be *quasi-homeomorphic* if X is Y-like and Y is X-like.

In this paper we establish the class of all compacta which are quasi-homeomorphic with a disk. Our main result is the following

THEOREM 1. *A compactum X is quasi-homeomorphic with a disk if and only if X is a 2-dimensional AR-space embeddable into E^2.*

First, we give an example of a compactum quasi-homeomorphic with a disk D which is not homeomorphic with D.

Example 1. Let X be a locally connected continuum and let X be the union of a sequence of disks $(D_i)_{i=1}^{\infty}$ such that $D_i \cap D_{i+1} = (p_i)$, $i = 1, 2, \ldots$, $\operatorname{diam}(D_i) \to 0$, and of a point q such that the space $X = \bigcup_{i=1}^{\infty} D_i \cup \{q\}$ is compact.

Obviously, X is a continuum which is not homeomorphic with D. Given an $\varepsilon > 0$, there is an index n_0 such that

$$\operatorname{diam}\Big(\bigcup_{i=n_0+1}^{\infty} D_i\Big) < \varepsilon.$$

We define a retraction r from X onto $\bigcup_{i=1}^{n_0} D_i$ as follows:

$$r\Big(\bigcup_{i=n_0+1}^{\infty} D_i\Big) = p_{n_0},$$

where p_{n_0} is a point in common of the disk D_{n_0} with D_{n_0+1}. For each p_i, $i = 1, \ldots, n_0 - 1$, choose arcs I_1^i and I_2^i such that

$$I_1^i \subset \dot{D}_i, \quad I_2^i \subset \dot{D}_{i+1}, \quad I_1^i \cap I_2^i = (p_i), \quad \text{and} \quad \operatorname{diam}(I_1^i \cup I_2^i) < \varepsilon/2.$$

Let us glue the disks along these arcs. Thus, we obtain an ε-mapping f_1 of $\bigcup_{i=1}^{n_0} D_i$ onto D. Then $f = f_1 r$ is an ε-mapping of X onto D.

It is easily seen that for any $\varepsilon > 0$ there is a subset X' of D homeomorphic with X and such that there is a retraction r of D onto X' being an ε-mapping. Thus there is an ε-mapping of D onto X.

2. Some lemmas. A locally connected continuum containing more than one point will be called *cyclic* if it is separated by no point. Given a locally connected continuum X, the following subsets of X will be called the *cyclic elements* of X: each point which separates X, each end-point of X (i.e., a point of order 1 in the sense of Menger-Urysohn), and each non-degenerate subset of X maximal with respect to the property of being a cyclic space. The graph K_1 is the 1-skeleton of a 3-simplex with mid-points of a pair of non-adjacent edges joined by a segment and the graph K_2 is the 1-skeleton of a 4-simplex. A 2-umbrella T is the one-point union of a disk and of an arc relative to an interior point of the disk and an end-point of the arc. Any set homeomorphic with S^2 is said to be a *simple surface*. A set is called *cyclic* if it is not disconnected by a point.

We shall base ourselves on the notion of a cyclic element and on the properties of cyclic elements given in [4].

Let X be a compactum quasi-homeomorphic with a disk.

2.1. *X is a locally connected continuum containing neither K_1 nor K_2, nor* T, *and X is not a simple surface.*

Since X is a continuous image of D, it is a locally connected continuum. Since X is D-like, X is quasi-embeddable into E^2. Then X does not contain K_1, K_2, T, and X is not a simple surface [6].

2.2. *No simple closed curve $S \subset X$ is a retract of X.*

By 2.1, X is a locally connected continuum. Suppose that there is a simple closed curve $S \subset X$ which is a retract of X. Consequently, the first homology group $H_1(X, Z)$ of X in the sense of Čech is not trivial, which yields a contradiction with Theorem 1 of [5] because $H_1(D, Z) = 0$.

2.3. *If E is a non-degenerate cyclic element of X, then no simple closed curve $S \subset E$ is a retract of E.*

Each cyclic element E of a locally connected continuum X is a retract of X (see [4]). If there is a simple closed curve $S \subset E$ which is a retract of E, then S is a retract of X, which is a contradiction by 2.2.

2.4. *If E is a cyclic element of X, then E is either a point or a disk.*

By 2.1, X is a locally connected continuum containing neither K_1 nor K_2, nor a 2-umbrella. It follows that X does not contain the simple surface S^2. Indeed, if X contains S^2, then $X - S^2 \neq \emptyset$. Since X is arcwise connected, X contains a 2-umbrella, which contradicts Lemma 2.1. Let E be a non-degenerate cyclic element of X. Then E is a locally connected continuum [4]. By 2.3, no simple closed curve $S \subset E$ is a retract of E. Moreover, E does not contain K_1, and E is not a simple surface. Consequently, E is a disk (cf. [7]).

2.5. Remark. In the proof of Lemma 2.4 we have shown that if E is a non-degenerate cyclic element of X, then E is a disk. Moreover, notice that X has a non-degenerate cyclic element, i.e., X contains a disk. Indeed, if X contains cyclic elements which are the points only, then X is a dendrite, which contradicts the assumption that X is quasi-homeomorphic with a disk (a dendrite is 1-dimensional).

2.6. *X is an AR embeddable into E^2.*

By 2.1 and 2.4, X is a locally connected continuum and each cyclic element of X is either a point or a disk. Notice that points and disks are the AR's. It follows that X is an AR (see [4], p. 346). By Theorem 3.1 in [9] and by 2.1, X is embeddable into E^2.

2.7. *If Y is an AR embeddable into E^2, then each cyclic element of Y is a point or a disk.*

This follows from [2] (p. 132) and [4] (p. 346 and p. 526).

2.8. *Let X, Y, and Z be compacta. If X is Y-like and Y is Z-like, then X is Z-like.*

Let $\varepsilon > 0$ be given. Since X is Y-like, there is an ε-mapping f_1 of X onto Y. By [4] (p. 35), for an ε-mapping f_1 of X onto Y there is an $\eta > 0$ such that if $B \subset Y$ and $\operatorname{diam} B < \eta$, then $\operatorname{diam} f_1^{-1}(B) < \varepsilon$. Since Y is Z-like, for an $\eta > 0$ there is an η-mapping f_2 of Y onto Z. Let $f = f_2 f_1$. Then it is clear that f is an ε-mapping of X onto Z.

3. Proof of Theorem 1. By 2.6 and [4] (§ 45, IV) the necessity of the conditions of Theorem 1 is clear, so it remains to prove the sufficiency. Assume that X satisfies the conditions of Theorem 1. Then, by 2.7, each cyclic element of X is a point or a disk.

Given an $\varepsilon > 0$, we shall first prove that there is an ε-mapping of D onto X. It is easy to see that for an $\varepsilon > 0$ there are a finite sequence $(a_i)_{i=1}^N$ of the boundary points of D and neighborhoods K_i of a_i in D. These neighborhoods have the diameter less than $\varepsilon/4$ and

$$0 < \varrho(a_i, a_{i+1}) = \eta < \varepsilon/4 \text{ for } i = 1, \ldots, N-1, \qquad 0 < \varrho(a_N, a_1) = \eta < \varepsilon/4,$$

$K_i \cap K_j = \emptyset$ for $i \neq j$, $\bar{K}_i \cap \bar{K}_{i+1} = (d_i)$ $(i = 1, \ldots, N-1)$, and $\bar{K}_N \cap \bar{K}_1 = (d_N)$, where d_i $(i = 1, \ldots, N)$ are the boundary points of D. As $\bar{K}_{i-1} \cup \bar{K}_i \cup \bar{K}_{i+1}$ $(i = 2, \ldots, N-1)$, $\bar{K}_{N-1} \cup \bar{K}_N \cup \bar{K}_1$, and $\bar{K}_N \cup \bar{K}_1 \cup \bar{K}_2$ are connected sets,

$$\begin{aligned}\operatorname{diam}(\bar{K}_{i-1} \cup \bar{K}_i \cup \bar{K}_{i+1}) &\leqslant \operatorname{diam} \bar{K}_{i-1} + \operatorname{diam} \bar{K}_i + \operatorname{diam} \bar{K}_{i+1} \\ &\leqslant \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4 \quad \text{for } i = 2, \ldots, N-1\end{aligned}$$

and, similarly,

$$\operatorname{diam}(\bar{K}_{N-1} \cup \bar{K}_N \cup \bar{K}_1) \leqslant 3\varepsilon/4 \quad \text{and} \quad \operatorname{diam}(\bar{K}_N \cup \bar{K}_1 \cup \bar{K}_2) \leqslant 3\varepsilon/4.$$

Indeed, for each i there is a retraction $r_i \colon \bar{K}_i \twoheadrightarrow L_i \cup L'_i$, where $L_i = \overline{a_i b_i}$, L'_i is the arc containing d_{i-1}, b_i, d_i, and $L_i \cap L'_i = b_i$ for $i = 2, \ldots, N$, and $L_1 = \overline{a_1 b_1}$, L'_1 is the arc containing d_N, b_1, d_1, $L_1 \cap L'_1 = b_1$ (Fig. 1). Let r be defined by

$$r(y) = \begin{cases} y & \text{if } y \in D' = D - \bigcup_{i=1}^{N} K_i, \\ r_i(y) & \text{if } y \in \bar{K}_i \ (i = 1, \ldots, N). \end{cases}$$

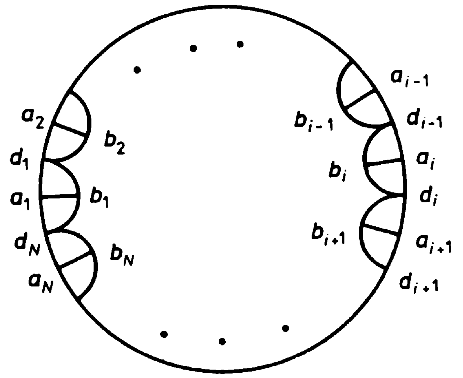

Fig. 1

Then r is a retraction of D onto $D' \cup \bigcup_{i=1}^{N} L_i$. Choose a non-degenerate cyclic element of X which is a disk D_0. Let h be a homeomorphism of D' onto D_0 and let $h(b_i) = x_i$. Let

$$Z = D' \cup \bigcup_{i=1}^{N} L_i.$$

Notice that the set of the components of $X - D_0$ is at most countable (see [4], p. 318) and the closure of each component of $X - D_0$ is a locally connected continuum having exactly one point in common with D_0 (cf. [4], p. 312). Now, we consider the set

$$A_i = x_i x_{i+1} \cup \bigcup_{k=1}^{\infty} \bar{C}^i_k, \quad i = 1, \ldots, N-1,$$

where $x_i x_{i+1} \subset \dot{D}_0$ and $\bar{C}^i_k$ denotes the closure of the component of $X - D_0$ which has only one point in common with $x_i x_{i+1}$ and

$$A_n = x_N x_1 \cup \bigcup_{k=1}^{\infty} \bar{C}^N_k.$$

Observe that the sequence $C^i_1, C^i_2, \ldots$ can be finite. It is easy to show that A_i is a locally connected continuum. Thus for each i there is a map $g_i\colon L_i \twoheadrightarrow A_i$. We can assume that $g_i(b_i) = x_i$. Consider the map $g'\colon Z \twoheadrightarrow X$ defined by

$$g'(z) = \begin{cases} h(z) & \text{if } z \in D', \\ g_i(z) & \text{if } z \in L_i \ (i = 1, \ldots, N). \end{cases}$$

Obviously, g' is continuous. Let $g = g'r$; then g is a map of D onto X. Let $x \in X$. If $x \in \operatorname{Int} D_0$, then $g^{-1}(x)$ is a point, and if $x \in X - \operatorname{Int} D_0$, then $x \in A_i$ for certain i. From the constructions of r and g' it follows that

$$g^{-1}(x) \subset \bar{K}_{i-1} \cup \bar{K}_i \cup \bar{K}_{i+1} \quad \text{for } i = 2, \ldots, N-1$$

and, similarly,

$$g^{-1}(x) \subset \bar{K}_{N-1} \cup \bar{K}_N \cup \bar{K}_1 \text{ if } x \in A_N, \quad g^{-1}(x) \subset \bar{K}_N \cup \bar{K}_1 \cup \bar{K}_2 \text{ if } x \in A_1.$$

Since

$$\operatorname{diam}(\bar{K}_{i-1} \cup \bar{K}_i \cup \bar{K}_{i+1}) < \varepsilon \quad \text{for } i = 2, \ldots, N-1,$$

$$\operatorname{diam}(\bar{K}_{N-1} \cup \bar{K}_N \cup \bar{K}_1) < \varepsilon \quad \text{and} \quad \operatorname{diam}(\bar{K}_N \cup \bar{K}_1 \cup \bar{K}_2) < \varepsilon,$$

we have $\operatorname{diam} g^{-1}(x) < \varepsilon$. Thus g is an ε-mapping of D onto X.

Now, let us prove that there is an ε-mapping of X onto D. In this proof we base ourselves on the notions of a set entirely arcwise connected and of a cyclic element and also on their properties. A subset A of a given locally connected continuum X is said to be *entirely arcwise connected* (*in* X) if $x, y \in A$ and $x \neq y$ imply that each arc (in X) joining x and y is contained in A.

We shall first show that the following additional assumption can be made.

3.1. *There is a finite set $F \subset X$ such that the least closed and entirely arcwise connected subset of X containing F is equal to X.*

Indeed, let F_i denote any finite subset of X such that for each point $x \in X$ there is a point $y \in F_i$ with $\varrho(x, y) < 1/i$, where $i = 1, 2, \ldots$ Let A_k denote the least closed and entirely arcwise connected subset of X containing the set $\bigcup_{i=1}^{k} F_i$. Assume that no A_k is equal to X. Then the sets A_k satisfy the assumptions of (3.14) in [8] and, therefore, there is an index k_0 such that the diameter of each component of $X - A_{k_0}$ is less than $\varepsilon/3$. Let r_0 denote the retraction of X onto A_{k_0} such that for each component C of $X - A_{k_0}$ we have

$r_0(\bar{C}) = \bar{C} - C$ (cf. [4], p. 346). Suppose that there is an $(\varepsilon/3)$-mapping f_0: $A_{k_0} \twoheadrightarrow D$. Then $f_0 r_0$ is an ε-mapping of X onto D, which implies that the additional assumption 3.1 can be made. (Notice that the set $A_{k_0} \subset X$ satisfies analogous assumptions to those satisfied by X.)

3.2. *There is an ε-mapping of X onto D.*

The proof of 3.2 will be inductive with respect to the number of m points of the set F mentioned in 3.1. Evidently, we can assume that $m > 1$. First, we consider the case $m = 2$. Let F consist of the points a_1 and a_2. By (3.13) in [8], X is the union of an arc L joining these points and of a sequence (finite or not) $E_1, E_2, \ldots$ of the non-degenerate cyclic elements of X (cf. [4]), where $E_i \cap L$ is a non-degenerate subarc L_i of L and $E_i \cap E_j = L_i \cap L_j = \dot{L}_i \cap \dot{L}_j$ for $i \neq j$. Notice that we can assume that the sequence $E_1, E_2, \ldots$ is finite. Otherwise, by [4] (p. 319) there is an index n_0 such that the diameter of each element E_n with $n > n_0$ is less than $\varepsilon/3$. Evidently, there is a retraction r_0 of X onto $L \cup \bigcup_{i=1}^{n_0} E_i$ such that $r_0(E_n) = L_n$ for $n > n_0$. The condition given in 3.2 will be satisfied if we consider the composition of r_0 and of a suitable map of $r_0(X)$ onto D.

Thus we can assume that X has exactly n_0 non-degenerate cyclic elements which are the disks (see 2.7). It is easy to see that X is the union of n_0 disks and m_0 arcs which can be ordered in a sequence $(A_i)_{i=1}^{n_0+m_0}$ such that $A_i \cap A_{i+1} = (x_i)$ and $A_i \cap A_j = \emptyset$ if $|i-j| > 1$. Then, by the same method as that used in Example 1 we can construct an ε-mapping of X onto D. Thus the proof of 3.2 for $m = 2$ is completed.

Induction step. Now, suppose that the set $F \subset X$ mentioned in 3.1 consists of m points, where $m > 2$, and assume that 3.2 is true for each space satisfying the assumptions analogous to those satisfied by X with the corresponding set having less than m points. Fix a point $a \in F$ and let Y denote the least closed and entirely arcwise connected subset of X containing the set $F-(a)$. We can assume that $a \notin Y$ and let C denote the component of $X - Y$ containing a. Then $\bar{C} - C$ consists of exactly one point b (cf. [4], p. 312). Let Z denote the least closed and entirely arcwise connected subset of X containing a and b. Then $Z \subset \bar{C}$. Since the set $Y \cup Z$ is closed and entirely arcwise connected (cf. [4], p. 313) and it contains F, we have $Y \cup Z = X$, whence $Z = \bar{C}$.

It is clear that the sets Y and Z satisfy the induction hypothesis and $Y \cap Z = (b)$. Therefore, there are two $(\varepsilon/2)$-mappings f_1: $Y \twoheadrightarrow D_1$ and f_2: $Z \twoheadrightarrow D_2$ such that $f_1(b) = f_2(b) = b'$ and $D_1 \cap D_2 = (b')$, where b' is a boundary point of disks D_1 and D_2. Then the function

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in Y, \\ f_2(x) & \text{if } x \in Z \end{cases}$$

is an ε-mapping of X onto $D_1 \cup D_2$. Since $D_1 \cup D_2$ is D-like (see Example 1), by 2.8 there is an ε-mapping of X onto D.

4. A characterization of the plane AR-spaces by means of quasi-homeomorphisms. A *dendrite* is any locally connected continuum which does not contain any simple closed curve. Notice that

4.1. *A dendrite is embeddable into E^2.*

4.2. *One-dimensional AR-spaces coincide with non-degenerate dendrites* (see [2], p. 138).

4.3. *A compactum quasi-homeomorphic with a dendrite is a dendrite.*

Since X is a continuous image of a dendrite, X is a locally connected continuum. By the theorem of Mardešić and Segal in [5], X does not contain a simple closed curve, whence X is a dendrite.

By 2.8 and Theorem 1 we obtain immediately the following

4.4. COROLLARY. *All plane 2-dimensional AR-spaces are quasi-homeomorphic.*

4.5. COROLLARY. *A compactum X is a plane AR if and only if X is quasi-homeomorphic either with a disk or with a dendrite.*

The necessity of the conditions follows from Theorem 1 and 4.2. The sufficiency follows from Theorem 1, 4.2 and 4.3.

By 2.8 and Corollary 4.5 we obtain

4.6. COROLLARY. *Each compactum quasi-homeomorphic with a plane AR is itself an AR embeddable into E^2.*

5. Compacta quasi-homeomorphic with a plane 2-dimensional manifold. We shall establish the class of all compacta which are quasi-homeomorphic with a plane compact 2-dimensional manifold, i.e., a disk with holes. By M we denote a plane compact 2-dimensional manifold which is not a disk. The main result of this section is the following

THEOREM 2. *A compactum X is quasi-homeomorphic with M if and only if X is a locally connected continuum embeddable into E^2, containing exactly one cyclic element disconnecting E^2, this cyclic element being homeomorphic with M.*

First, we consider two examples.

Example 2. Let M be a disk with two holes, and let X be a disk with two holes such that the closures of these holes have one point x_0 in common.

We show that M is not X-like. Let C_i, C_i', S_i, S_i' denote, respectively, the components of $E^2 - M$, the components of $E^2 - X$, the boundaries of C_i, and the boundaries of C_i' for $i = 1, 2, 3$ (Fig. 2). The boundaries S_i and S_i' are simple closed curves such that $S_i \subset M$ and $S_i' \subset X$. Take an $\varepsilon \in (0, \eta/3]$, where

$$\eta = \min(\varrho(S_1, S_2), \varrho(S_2, S_3), \varrho(S_1, S_3)) > 0.$$

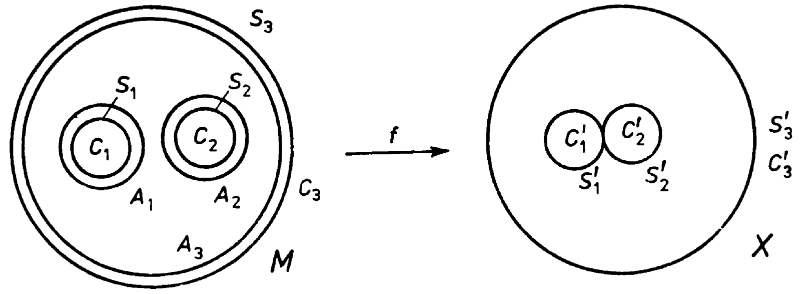

Fig. 2

First, we shall show the following fact:

5.1. *If f is an ε-mapping of M onto X, then for each point $a \in M$ such that $\varrho(a, S_i) > \varepsilon$ for $i = 1, 2, 3$ we have $f(a) \in \operatorname{Int} X$.*

If $a_0 \in M$ and $\varrho(a_0, S_i) > \varepsilon$ for $i = 1, 2, 3$, then the set

$$U = \{a \in E^2 \colon \varrho(a_0, a) < \varepsilon\}$$

is contained in M. By a result of Borsuk [1], there is an open set $V \subset E^2$ (also in X) such that $f(a_0) \in V \subset f(M) = X$. Consequently, $f(a_0) \in \operatorname{Int} X$.

Now, assume that M is X-like and let f be an ε-mapping of M onto X. Put

$$A_i = \{a \in M \colon \varrho(a, S_i) \leqslant \varepsilon\}.$$

By 5.1 we have

$$\bigcup_{i=1}^{3} f^{-1}(S_i') \subset \bigcup_{i=1}^{3} A_i.$$

Now, we consider the set $f^{-1}(S_1')$. Let x be a point of S_1' and let m be a point of M such that $f(m) = x$; then $m \in \bigcup_{i=1}^{3} A_i$. We can assume that $m \in A_1$. Let m' be any point of M such that $f(m') = x$. Since $\varrho(A_i, A_j) \geqslant \varepsilon$ for $i \neq j$ $(i, j = 1, 2, 3)$, we have $m' \in A_1$. Thus, for any point $x \in S_1'$ there is an index $i_x \leqslant 3$ such that $f^{-1}(x) \subset A_{i_x}$. Since f is an ε-mapping and $\varrho(A_i, A_j) \geqslant \varepsilon$ for $i \neq j$, the set $\{x \in S_1' \colon f^{-1}(x) \subset A_i\}$ is closed in S_1' for each i and these sets are disjoint. Then from the connectivity of S_1' we infer that $f^{-1}(x) \subset A_1$ for every $x \in S_1'$.

Now, observe that the inclusion $f^{-1}(S_1') \cup f^{-1}(S_2') \subset A_1$ is impossible. Indeed, if the inclusion holds true, then $f(A_1)$ is a continuum (locally connected) containing $S_1' \cup S_2'$. If the set $f^{-1}(S_3')$ is contained, say, in A_2, then $f(A_2)$ is a continuum containing S_3'. It is easily seen that the sets $f(A_i)$ are disjoint, and hence the set $f(A_3)$ lies in the interior of X. By Fort's lemma (see [3]), the image $f(S_3) \subset f(A_3)$ contains a simple closed curve being a retract of X, therefore disconnecting X between S_3' and $S_1' \cup S_2'$.

Then the set $f(M - \bigcup_{i=1}^{3} A_i)$ intersects the set $f(S_3)$ or it is a disconnected set. In both cases we obtain a contradiction.

Now, we consider the point x_0. Since $x_0 \in S_1' \cap S_2'$, we have $f^{-1}(x_0) \subset A_1$ and $f^{-1}(x_0) \subset A_2 \cup A_3$, which contradicts the fact that A_i's are disjoint.

Example 3. Let M be a disk with two holes and let $X = M_1 \cup M_2$, where M_1 and M_2 are disks with one hole such that $M_1 \cap M_2 = (x_0)$ and x_0 is a boundary point of M_1 and of M_2.

We shall show that M is also not X-like. Let S_i (S_i') denote the boundaries of the components of $E^2 - M$ (of $E^2 - X$) for $i = 1, 2, 3$. The boundaries S_i for $i = 1, 2, 3$ and S_1', S_2' are simple closed curves, and S_3' is the union of two simple closed curves with one common point x_0, where $S_i \subset M$ and $S_i' \subset X$. Take ε, η, and A_i as in Example 2. Now, assume that M is X-like and let f be an ε-mapping of M onto X. By the same argument as in Example 2 it follows from 5.1 that each set $f^{-1}(S_i')$ is contained in one of the sets A_i $(i = 1, 2, 3)$ and that the inclusion $f^{-1}(S_1') \cup f^{-1}(S_2') \subset A_1$ is impossible. Let us consider the connected set $M - A_3$. Suppose that $f^{-1}(S_1') \subset A_1$ and $f^{-1}(S_2') \subset A_2$; then $f^{-1}(S_3') \subset A_3$ and $S_1' \cup S_2' \subset f(M - A_3)$, where $S_1' \subset M_1$, $S_2' \subset M_2$, $M_1 \cap M_2 = (x_0)$. Since $x_0 \in S_3'$, we have $x_0 \notin f(M - A_3)$. Consequently, the set $f(M - A_3)$ is not connected, which yields a contradiction.

Proof of Theorem 2. Let X be a compactum quasi-homeomorphic with M, where M has exactly $n \geqslant 1$ holes. Then:

5.2. *X is a locally connected continuum embeddable into E^2 and $E^2 - X$ has n bounded components.*

5.3. *If E is a cyclic element of X, then E is either a point or a disk, or a disk with holes where the number of holes of E is not greater than n.*

Since X is a continuous image of M, X is a locally connected continuum. By Theorem 3.1 in [9], X is embeddable into E^2. Since X is quasi-homeomorphic with M, $E^2 - X$ has exactly n bounded components [10].

Let E be a non-degenerate cyclic element of X; then E is a locally connected cyclic continuum embeddable into E^2. Assume that E is not a disk. Then E disconnects E^2. Let C_i $(i = 1, \ldots, k)$ denote all bounded components of $E^2 - E$. Evidently, $k \leqslant n$. Then $\bar{C}_i$ is a disk and $\bar{C}_i \cap E = \mathrm{Bd}(\bar{C}_i)$ is a simple closed curve ([4], p. 506). Let $\tilde{E}$ denote the union of E and of $\bigcup_{i=1}^{k} \bar{C}_i$. Since $\mathrm{Bd}(\bar{C}_i) \subset E$, we have

$$\tilde{E} = E \cup \bigcup_{i=1}^{k} C_i.$$

$\tilde{E}$ is a locally connected continuum and, moreover, $\tilde{E}$ is a disk ([4], p. 526). We can prove that the boundaries of C_i cannot have common points and, therefore, E is a manifold.

By the same method as in Example 3 we infer that:

5.4. *X has exactly one cyclic element E, being a disk with n holes and, therefore, E is homeomorphic with M.*

By 5.2 and [4] (§ 45.IV) we see that the conditions given in Theorem 2 are necessary. Using the same argument as in the proof of Theorem 1 we can show that X is quasi-homeomorphic with M, which proves that the conditions are also sufficient.

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