COLLOQUIUM MATHEMATICUM

VOL. XXXII 1975 FASC. 2

REAL INTEGRABLE SPACES

BY

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1. Introduction. In [13] an *integrable space* was defined as a pair $\mathscr{L} = (L, \nu)$, where L is a group of functions and ν is a kind of a "countably subadditive" seminorm on L. Some basic properties of $\mathscr{L}$ and its closure-type extension $\overline{\mathscr{L}}$ were established therein. The investigation of integrable spaces was continued in [14]. The present paper is also a continuation of [13], but it goes in another direction. Namely, we restrict ourselves to the case where L is a Riesz subgroup of R^S, thus arriving at the notion of a real integrable space (Definition 2). We study some properties of such spaces connected with the continuity of ν with respect to various kinds of pointwise convergence (Fatou, Daniell, saturability, Lebesgue and Beppo Levi properties). They have been mainly suggested by the theory of Lebesgue integration. We prove that all these properties are preserved when $\mathscr{L}$ is extended to $\overline{\mathscr{L}}$ (Theorems 1-4 and 8). As an application we give a new proof of the remarkable Pellaumail's extension theorem for Daniell spaces [10] (Theorem 6 in Section 4).

2. Preliminaries and Fatou property. We adopt here the terminology and notation introduced in [13]. In particular, S denotes an abstract set, $(X, |\cdot|)$ stands for an abelian complete normed group, and $\mathscr{L} = (L, \nu)$ is an integrable space, where $L \subset X^S$ (see [13], Definition 1). We start with establishing some elementary properties of integrable spaces, which are not pointed out in [13].

PROPOSITION 1. *If $\{f_n\} \subset \overline{L}$ is a Cauchy sequence in L^* and*

$$\lim_{n\to\infty} f_n = f,$$

then

$$f \in \overline{L} \quad \text{and} \quad \lim_{n\to\infty} N(f_n - f) = 0.$$

Here and in what follows by

$$\lim_{n\to\infty} f_n = f$$

we mean that

$$\lim_{n\to\infty} f_n(s) = f(s) \quad \text{for all } s \in S.$$

Proof. Choose a subsequence $\{f_{n_m}\}$ with

$$\sum_{m=1}^{\infty} N(f_{n_{m+1}} - f_{n_m}) < \infty.$$

Since

$$f(s) - f_{n_1}(s) = \sum_{m=1}^{\infty} \big(f_{n_{m+1}}(s) - f_{n_m}(s)\big) \quad \text{for } s \in S,$$

we have

$$f - f_{n_1} \overset{\overline{\mathscr{L}}}{\sim} \sum_m (f_{n_{m+1}} - f_{n_m}).$$

Hence, by Corollary 1(3) of [13],

$$f \in \bar{L} \quad \text{and} \quad \lim_{k\to\infty} N\Big(f - f_{n_1} - \sum_{m=1}^{k} (f_{n_{m+1}} - f_{n_m})\Big) = 0.$$

Thus

$$\lim_{k\to\infty} N(f - f_{n_k}) = 0.$$

COROLLARY 1. *Let $\{f_n\} \subset \bar{L}$ be a Cauchy sequence in L^* and let $f \in R^S$. Then the following two conditions are equivalent:*

(i) $f \in \bar{L}$ *and* $\lim_{n\to\infty} N(f_n - f) = 0$.

(ii) *There are a subsequence $\{f_{n_m}\}$ and a sequence $\{g_m\} \subset \bar{L}$ such that*

$$\lim_{m\to\infty} g_m = f \quad \text{and} \quad N(f_{n_m} - g_m) = 0, \quad m = 1, 2, \ldots$$

Proof. Assume that (i) holds. Then there exists a subsequence $\{f_{n_m}\}$ with

$$\sum_{m=1}^{\infty} N(f_{n_m} - f) < \infty.$$

Put $g_m(s) = f_{n_m}(s)$ if $\sum_{k=1}^{\infty} |f_{n_k}(s) - f(s)| < \infty$ and $g_m(s) = f(s)$ otherwise. Since

$$|f_{n_m} - g_m| \leqslant \sum_{k=l}^{\infty} |f_{n_k} - f| \quad \text{for } l = 1, 2, \ldots,$$

it follows that $g_m \in L^*$ ($L^* = L^{**}$) and $N(f_{n_m} - g_m) = 0$.

Assume that (ii) holds. Then $\{g_m\}$ is a Cauchy sequence in L^*, so that, by Proposition 1,

$$f \in \bar{L} \quad \text{and} \quad \lim_{m\to\infty} N(g_m - f) = 0.$$

It follows that

$$\lim_{n\to\infty} N(f_n - f) = 0.$$

Definition 1 (cf. [8], p. 42). An integrable space $\mathscr{L}$ has the *Fatou property* if

$$\text{(F)} \qquad |f| \leqslant \lim_{n\to\infty}\uparrow |f_n| \text{ implies } \nu(|f|) \leqslant \lim_{n\to\infty}\uparrow \nu(|f_n|) \ (f, f_n \in L).$$

We use the symbol $\lim\uparrow$ ($\lim\downarrow$) to indicate that the convergence we deal with is non-decreasing (non-increasing). The limit values $+\infty$ or $-\infty$ are not excluded.

PROPOSITION 2. *Suppose $\mathscr{L}$ has the Fatou property and $|L| + |L| \subset |L|$. Then, for every $f \in \bar{L}$,*

$$\bar{\nu}(|f|) = \inf\{\lim_{n\to\infty}\uparrow \nu(|f_n|) \colon \{f_n\} \subset L \ \& \ |f| \leqslant \lim_{n\to\infty}\uparrow |f_n|\}.$$

Proof. Put

$$N'(f) = \inf\{\lim_{n\to\infty}\uparrow \nu(|f_n|) \colon \{f_n\} \subset L \ \& \ |f| \leqslant \lim_{n\to\infty}\uparrow |f_n|\} \qquad \text{for } f \in L^*.$$

By the assumption that $|L| + |L| \subset |L|$, $N' \leqslant N$ and N' is a semi-norm on L^*. The Fatou property implies that $N'(f) = N(f)$ if $f \in L$. The continuity of N' with respect to the topology generated by N now yields $N'(f) = N(f)$ for all $f \in \bar{L}$.

Suppose that L is a subgroup of X^S satisfying $|L| + |L| \subset |L|$, and $\nu \colon |L| \to R^+$ is such that $\nu(0) = 0$ and $\nu(|f| + |g|) \leqslant \nu(|f|) + \nu(|g|)$ (subadditivity). Then $\mathscr{L} = (L, \nu)$ is an integrable space provided that (F) holds. The following example shows that the converse statement is not true.

Example 1. Let $X = R$ and put $L = \{a\chi_S \colon a \in R\}$. Define $\nu(0) = 0$, $\nu(a\chi_S) = 1$ for $a \in (0, 1)$ and $\nu(a\chi_S) = 2$ for $a \in [1, \infty)$. Clearly, the pair (L, ν) is an integrable space, but (F) does not hold.

3. Daniell, saturability and Lebesgue properties. In the following $(R, +, \leqslant, |\cdot|)$ stands for the additive group of the real numbers with the usual ordering and the euclidean norm $|\cdot|$. Given a (non-empty) set S, the pointwise addition and partial ordering make R^S into a lattice-ordered group. The joint and meet in R^S are denoted by $\vee$ and $\wedge$, respectively. For any $F \subset R^S$, we write $F^+ = \{f^+ \colon f \in F\}$.

A family $F \subset R^S$ is said to be a *Riesz subgroup* of R^S if it is a subgroup and a sublattice of R^S.

Remark 1. A subgroup F of R^S is a Riesz subgroup of R^S if and only if $F^+ \subset F$. In this case $|F| = F^+ = \{f \in F \colon f \geqslant 0\}$.

Definition 2. A pair $\mathscr{L} = (L, \nu)$ is a real integrable space if

(a) L is a Riesz subgroup of R^S,

(b) ν is a function from L^+ into R^+ such that $\nu(0) = 0$ and if

$$f, f_n \in L^+ \quad \text{and} \quad f \leqslant \sum_{n=1}^{\infty} f_n,$$

then

$$\nu(f) \leqslant \sum_{n=1}^{\infty} \nu(f_n).$$

Clearly, any real integrable space is an integrable space in the sense of [13], Definition 1. Suppose $\mathscr{L} = (L, \nu)$ is an integrable space with $L \subset R^S$. Then $\mathscr{L}$ is real if and only if L is a sublattice of R^S. In particular, $\mathscr{L}^* = (L^*, \nu^*)$ is a real integrable space.

Throughout the rest of the paper $\mathscr{L} = (L, \nu)$ denotes an arbitrary real integrable space.

PROPOSITION 3. *$\overline{\mathscr{L}} = (\bar{L}, \bar{\nu})$ is a real integrable space and $\bar{L}^+ \subset \overline{L^+}$.*

Proof. By Remark 1, it is enough to prove the second part of the assertion. Put $\varphi(f) = f^+$ for $f \in L^*$. The inequality $|f^+ - g^+| \leqslant |f - g|$ shows that φ is a continuous function on L^*. Hence $\varphi(\bar{L}) \subset \overline{\varphi(L)}$.

The following lemma will be a useful tool in our further investigations:

LEMMA 1. *If $\{f_n\}$ is a non-decreasing sequence in $\bar{L}$, then, given an $\varepsilon > 0$, there exists a non-decreasing sequence $\{g_n\}$ in L such that*

$$\lim_{n\to\infty} \uparrow f_n \leqslant \lim_{n\to\infty} \uparrow g_n \quad \text{and} \quad N(f_n - g_n) \leqslant \varepsilon \quad \text{for } n = 1, 2, \ldots$$

The functions g_n can be chosen non-negative (non-positive) provided that f_n are non-negative (non-positive).

Proof. By the definitions of L^* and $\bar{L}$ (see [13], p. 911-912), there exist $h_n \in L$ and $h_{nm} \in L^+$ such that

$$|f_n - h_n| \leqslant \sum_{m=1}^{\infty} h_{nm} \quad \text{and} \quad \sum_{m=1}^{\infty} \nu(h_{nm}) \leqslant \varepsilon \cdot 2^{-n-1} \quad (n = 1, 2, \ldots).$$

Write

$$g_k = \bigvee_{n=1}^{k} h_n + \sum_{n,m=1}^{k} h_{nm} \quad \text{for } k = 1, 2, \ldots$$

Clearly, the sequence $\{g_k\} \subset L$ is non-decreasing and

$$f_n \leqslant \lim_{k\to\infty} \uparrow g_k \quad \text{for } n = 1, 2, \ldots$$

Moreover, we have

$$|f_k - g_k| = \Big|\bigvee_{n=1}^{k} f_n - \bigvee_{n=1}^{k} h_n - \sum_{n,m=1}^{k} h_{nm}\Big| \leqslant \sum_{n=1}^{k} |f_n - h_n| + \sum_{n,m=1}^{k} h_{nm} \leqslant 2 \sum_{n,m=1}^{\infty} h_{nm},$$

so that

$$N(f_k-g_k) \leqslant 2\sum_{n=1}^{\infty}\varepsilon\cdot 2^{-n-1} = \varepsilon \qquad (k = 1, 2, \ldots).$$

To prove the second part of the assertion, replace g_n by g_n^+ (or by $-g_n^-$).

THEOREM 1. *If $\mathscr{L}$ has the Fatou property, then $\overline{\mathscr{L}}$ also has this property.*

Proof. Assume that

$$f, f_n \in \bar{L}^+ \quad \text{and} \quad f \leqslant \lim_{n\to\infty}\uparrow f_n.$$

Given $\varepsilon > 0$, take a sequence $\{g_n\} \subset L^+$ as in Lemma 1. Since

$$f \leqslant \lim_{n\to\infty}\uparrow g_n,$$

Proposition 2 shows that

$$\bar{\nu}(f) \leqslant \lim_{n\to\infty}\uparrow \nu(g_n).$$

This implies

$$\bar{\nu}(f) \leqslant \lim_{n\to\infty}\uparrow \bar{\nu}(f_n) + \varepsilon.$$

Hence

$$\bar{\nu}(f) \leqslant \lim_{n\to\infty}\uparrow \bar{\nu}(f_n).$$

Definition 3. A real integrable space $\mathscr{L}$ has the *Daniell property* if

(D) $$\lim_{n\to\infty}\downarrow f_n = 0 \text{ implies } \lim_{n\to\infty}\downarrow \nu(f_n) = 0 \quad (f_n \in L^+).$$

Since, for $f, f_n \in L^+$,

$$\nu(f) \leqslant \nu(f_n) + \nu\big((f-f_n)^+\big) \quad \text{and} \quad \lim_{n\to\infty}\downarrow (f-f_n)^+ = 0$$

provided that

$$f \leqslant \lim_{n\to\infty}\uparrow f_n,$$

the Daniell property implies the Fatou property. The converse fails to be true in general.

Example 2. Let L be a Riesz subgroup of R^S consisting of bounded functions and put

$$\nu(f) = \sup_{s\in S} f(s) \qquad \text{for } f \in L^+.$$

Then (L, ν) is a real integrable space having the Fatou property. Obviously, condition (D) need not hold. Note that $\bar{L}$ is the uniform closure of L and

$$\bar{\nu}(g) = \sup_{s\in S} g(s) \qquad \text{for all } g \in \bar{L}^+.$$

(This follows from the inequality $\nu^*(f) \geqslant \sup_{s\in S} f(s)$ for $f \in (L^*)^+$ and Proposition 1.) Moreover, if L contains all constant functions on S, then L^* is precisely the family of all bounded real functions on S and

$$\nu^*(f) = \sup_{s\in S} f(s) \quad \text{for all } f \in (L^*)^+,$$

so that the seminorm N generates the topology of the uniform convergence on L^*.

The following remark contains a strengthening of the Daniell property:

Remark 2. *Suppose $\mathscr{L}$ has the Daniell property. Then*

$$\lim_{n\to\infty}\downarrow f_n \leqslant f \textit{ implies } \lim_{n\to\infty}\downarrow \nu(f_n) \leqslant \nu(f) \quad (f_n, f \in L^+).$$

Proof. Since

$$\lim_{n\to\infty}\downarrow (f_n - f)^+ = 0$$

provided that

$$\lim_{n\to\infty}\downarrow f_n \leqslant f,$$

the assertion follows from the inequality $f_n \leqslant f + (f_n - f)^+$.

THEOREM 2. *If $\mathscr{L}$ has the Daniell property, then $\bar{\mathscr{L}}$ also has this property.*

Proof. Suppose that

$$f_n \in \bar{L}^+ \quad \text{and} \quad \lim_{n\to\infty}\downarrow f_n = 0,$$

and fix $\varepsilon > 0$. We deduce from Lemma 1, applied to $\{-f_n\}$, that there exists a sequence $\{g_n\} \subset L^+$ with

$$\lim_{n\to\infty}\downarrow g_n = 0 \quad \text{and} \quad N(f_n - g_n) \leqslant \varepsilon \quad (n = 1, 2, \ldots).$$

Since

$$\lim_{n\to\infty}\downarrow \nu(g_n) = 0,$$

we get

$$\lim_{n\to\infty}\downarrow \bar{\nu}(f_n) \leqslant \varepsilon.$$

PROPOSITION 4. *Suppose $\mathscr{L}$ has the Daniell property. Then, for every $f \in \bar{L}^+$,*

$$\bar{\nu}(f) = \sup\left\{\lim_{n\to\infty}\downarrow \nu(f_n) \colon \{f_n\} \subset L^+ \;\&\; \lim_{n\to\infty}\downarrow f_n \leqslant f\right\}.$$

Proof. The inequality $\geqslant$ follows from Remark 2 and Theorem 2.

To prove the converse one, put $f_n = -f$ and fix $\varepsilon > 0$. Lemma 1, applied to the sequence $\{f_n\}$ and ε, yields a sequence $\{g_n\} \subset L$ with

$$-f \leqslant \lim_{n\to\infty} \uparrow g_n \leqslant 0 \quad \text{and} \quad N(f+g_n) \leqslant \varepsilon.$$

Putting $h_n = -g_n$, we obtain

$$\{h_n\} \subset L^+, \quad \lim_{n\to\infty} \downarrow h_n \leqslant f \quad \text{and} \quad \bar{\nu}(f) \leqslant \lim_{n\to\infty} \downarrow \nu(h_n) + \varepsilon.$$

Definition 4 (cf. [12], p. 158, and [10], p. 1228). A real integrable space $\mathscr{L}$ has the *saturability property* if

$$\text{(S)} \qquad \sum_{n=1}^{\infty} f_n \leqslant f \text{ implies } \lim_{n\to\infty} \nu(f_n) = 0 \ (f_n, f \in L^+).$$

A sequence $\{f_n\} \subset F$, where F is a Riesz subgroup of R^S, is said to be *order-bounded* in F if there exists an $f \in F$ with $|f_n| \leqslant f$ for $n = 1, 2, \ldots$

PROPOSITION 5. *$\mathscr{L}$ has the saturability property if and only if every monotone order-bounded sequence in L is a Cauchy sequence in L^*.*

Proof. Necessity. Assume, to get a contradiction, that $\{f_n\} \subset L$ is a non-decreasing sequence which is not Cauchy in L^*, and that $|f_n| \leqslant f \in L$ for $n = 1, 2, \ldots$ There are an $\varepsilon > 0$ and a subsequence $\{f_{n_m}\}$ with $N(f_{n_{m+1}} - f_{n_m}) > \varepsilon$ for $m = 1, 2, \ldots$ Put $g_m = f_{n_{m+1}} - f_{n_m}$ for $m = 1, 2, \ldots$ We have, for $m = 1, 2, \ldots$,

$$\{g_m\} \subset L^+, \quad \sum_{m=1}^{\infty} g_m = \lim_{m\to\infty} f_{n_m} - f_{n_1} \leqslant f - f_{n_1} \in L^+ \quad \text{and} \quad \nu(g_m) > \varepsilon$$

which is impossible.

Sufficiency. If

$$f, f_n \in L^+ \quad \text{and} \quad \sum_{n=1}^{\infty} f_n \leqslant f,$$

then $\left\{\sum_{n=1}^{m} f_n\right\}$ is a non-decreasing order-bounded sequence in L; thus it is a Cauchy sequence in L^*. It follows that

$$\lim_{n\to\infty} \nu(f_n) = 0.$$

As an immediate consequence of Propositions 5 and 1 we get

COROLLARY 2. *Let $\mathscr{L}$ have the saturability property. If $\{f_n\}$ is a monotone order-bounded sequence in L with*

$$\lim_{n\to\infty} f_n = f,$$

then

$$f \in \bar{L} \quad \textit{and} \quad \lim_{n\to\infty} N(f_n - f) = 0.$$

In particular, if $\mathscr{L}$ has the saturability property, then it has the Daniell property. The converse assertion need not hold.

Example 3. In Example 2 take for S a topological space and for L the family of all continuous real functions on S with compact support. Dini's theorem shows that the integrable space (L, ν) has the Daniell property. Suppose S is a locally compact Hausdorff space. If S is non--discrete, then one can construct an infinite sequence of non-empty disjoint open sets contained in a compact subset of S, and, consequently, choose f_n, $f \in L^+$ with $\sum_{n=1}^{\infty} f_n \leqslant f$ and $\nu(f_n) = 1$ $(n = 1, 2, \ldots)$. Therefore, (L, ν) has the saturability property if and only if S is discrete.

THEOREM 3. *If $\mathscr{L}$ has the saturability property, then $\overline{\mathscr{L}}$ also has this property.*

Proof. Let $\{f_n\}$ be a non-decreasing sequence in $\bar{L}$ with $f_n \leqslant f \in \bar{L}$ for $n = 1, 2, \ldots$ Given $\varepsilon > 0$, choose a $g \in L$ with $N(f-g) < \varepsilon/6$ and a non-decreasing sequence $\{g_n\}$ in L with $N(f_n - g_n) < \varepsilon/6$ for $n = 1, 2, \ldots$ (see Lemma 1). It follows from the inequality

$$|f_n - g_n \wedge g| \leqslant |f_n - g_n| + |f - g|$$

that $N(f_n - g_n \wedge g) < \varepsilon/3$. Since $\{g_n \wedge g\}$ is a monotone order-bounded sequence in L, by Proposition 5, there is an n_0 such that

$$N(g_n \wedge g - g_m \wedge g) < \varepsilon/3 \quad \text{for all } n, m \geqslant n_0.$$

Hence $N(f_n - f_m) < \varepsilon$ for $n, m \geqslant n_0$. Thus $\{f_n\}$ is a Cauchy sequence in L^*, and the assertion follows from Proposition 5.

Definition 5. A real integrable space $\mathscr{L}$ has the *Lebesgue property* if

(L) $|f_n| \leqslant g$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} f_n = f$ imply

$f \in L$ and $\lim_{n\to\infty} \nu(|f_n - f|) = 0$ $(f_n, g \in L)$.

Clearly, the Lebesgue property implies the saturability property, but not conversely (see the Example in [13] with $X = R$; cf. also Proposition 6 below).

Definition 6 (cf. [7], p. 149). A subfamily F of R^S is said to be *σ-reticulated* if the pointwise limit of any non-decreasing (non-increasing) upper (lower) bounded sequence in F belongs to F.

Since

$$\lim_{n\to\infty} f_n = f \text{ implies } \lim_{n\to\infty}\downarrow \lim_{m\to\infty}\uparrow \bigvee_{k=n}^{n+m} f_k = f,$$

we have

Remark 3. A Riesz subgroup F of R^S is σ-reticulated if and only if the pointwise limit of any convergent order-bounded sequence in F belongs to F.

PROPOSITION 6. *$\mathscr{L}$ has the Lebesgue property if and only if it has the Daniell property and L is σ-reticulated.*

Proof. The necessity follows from Remark 3.

Sufficiency. Suppose f_n, g, f are as in condition (L). By Remark 3, $f \in L$. Put

$$g_n = \lim_{m\to\infty} \uparrow \bigvee_{k=n}^{n+m} |f_n - f_k| \quad \text{for } n = 1, 2, \ldots$$

Clearly,

$$\lim_{n\to\infty} \downarrow g_n = 0.$$

Moreover, since L is σ-reticulated and $g_n \leqslant g+g$, we have $g_n \in L$. Hence

$$\lim_{n\to\infty} \downarrow \nu(g_n) = 0.$$

The inequality $|f_n - f| \leqslant g_n$ yields

$$\lim_{n\to\infty} \nu(|f_n - f|) = 0.$$

THEOREM 4. *For a real integrable space $\mathscr{L}$ the following four conditions are equivalent:*

(i) *$\mathscr{L}$ has the Daniell property and $\bar{L}$ is σ-reticulated;*

(ii) *$\mathscr{L}$ has the saturability property;*

(iii) $|f_n| \leqslant f$ $(n = 1, 2, \ldots)$ *and* $\lim_{n\to\infty} f_n = 0$ *imply*

$$\lim_{n\to\infty} \nu(|f_n|) = 0 \qquad (f_n, f \in L)$$

(Banach's property; see [2], p. 322);

(iv) *$\bar{\mathscr{L}}$ has the Lebesgue property.*

Proof. It follows from Theorem 2 and Proposition 6 that (i) implies (iv). Obviously, (iv) implies (iii), which, in turn, implies (ii). Finally, by Theorem 3 and Corollary 2, (ii) implies (i).

Given a family $F \subset R^S$, we denote by F^σ the smallest σ-reticulated Riesz subgroup of R^S containing F.

Let $\mathscr{L} = (L, \nu)$ be a real integrable space such that $L^\sigma \subset \bar{L}$. Put $\mathscr{L}^\sigma = (L^\sigma, \nu^\sigma)$, where $\nu^\sigma = \bar{\nu}\,|\,(L^\sigma)^+$. Clearly, $\mathscr{L}^\sigma$ is a real integrable space. From Theorem 4 and Remark 3 we obtain

COROLLARY 3. *If $\mathscr{L}$ has the saturability property, then $\mathscr{L}^\sigma$ has the Lebesgue property.*

The next theorem establishes a kind of minimality of $\mathscr{L}^\sigma$ in the family of all real integrable spaces extending $\mathscr{L}$ and having the Lebesgue property. The proof of this theorem is based upon the following lemma, which is an analogue of the theorem on the smallest monotone class containing a ring of sets:

LEMMA 2. *If F is a Riesz subgroup of R^S, then F^σ is the smallest σ-reticulated subfamily of R^S containing F.*

Proof. Let $G \subset R^S$ be the smallest σ-reticulated family containing F, and put $H(f) = \{h \in R^S \colon f-h, h-f, f \vee h \in G\}$ for $f \in R^S$. Clearly, $h \in H(f)$ if and only if $f \in H(h)$. Moreover, as easily seen, $H(f)$ is σ-reticulated. Finally, $F \subset H(f)$ for $f \in F$. It follows that $G \subset H(f)$ for $f \in F$. Hence $F \subset H(g)$ for $g \in G$, so that $G \subset H(g)$ for $g \in G$. This shows that G is a Riesz subgroup of R^S. Thus $G = F^\sigma$.

THEOREM 5. *Let $\mathscr{L}_0 = (L_0, \nu_0)$ be a real integrable space having the Lebesgue property. If $\mathscr{L} = (L, \nu)$ is a real integrable space such that $L \subset L_0$ and $\nu = \nu_0 | L^+$, then $L^\sigma \subset L_0$ and $\nu^\sigma = \nu_0 | (L^\sigma)^+$, where $\mathscr{L}^\sigma = (L^\sigma, \nu^\sigma)$ is as in Corollary 3.*

Proof. In view of Remark 3, L_0 is σ-reticulated, so that $L^\sigma \subset L_0$. Since $\mathscr{L}$ has the saturability property, $\mathscr{L}^\sigma$ has the Lebesgue property by Corollary 3. Put

$$G = \{f \in L^\sigma \colon \nu^\sigma(|f|) = \nu_0(|f|)\}.$$

By assumption, $G \supset L$. Moreover, G is σ-reticulated. It follows from Lemma 2 that $G = L^\sigma$.

4. Daniell spaces. The main aim of this section is to give a new proof of Pellaumail's extension theorem for Daniell spaces [10] (see also Theorem 6 below). Our proof is based upon some properties of real integrable spaces established in Section 3.

We begin with the following

LEMMA 3. *Let F be a Riesz subgroup of R^S. If*

$$g, f_n \in F^+ \quad \text{and} \quad g \leqslant \sum_{n=1}^{\infty} f_n,$$

then there are $g_n \in F^+$ such that

$$g = \sum_{n=1}^{\infty} g_n \quad \text{and} \quad g_n \leqslant f_n \quad \text{for } n = 1, 2, \ldots$$

Proof. Put $g_1 = g \wedge f_1$ and

$$g_n = \Big(g \wedge \sum_{k=1}^{n} f_k\Big) - \Big(g \wedge \sum_{k=1}^{n-1} f_k\Big) \quad \text{for } n = 2, 3, \ldots$$

Let $(Y, |\cdot|)$ be an abelian seminormed group. Consider a pair (L, I), where L is a Riesz subgroup of R^S and $I\colon L \to Y$ is additive (i. e., $I(f+g) = I(f)+I(g)$ for all $f, g \in L$). Put

$$\nu(f) = \sup\{|I(g)|\colon g \in L^+ \ \&\ g \leqslant f\} \quad \text{for } f \in L^+.$$

Clearly, $\nu(0) = 0$, ν is monotone and subadditive. Up to Remark 5 below we additionally assume that $\nu(f)$ is finite for all $f \in L^+$. (This is trivially satisfied if $|\cdot|$ is bounded.)

Remark 4. We have $|I(f)| \leqslant 2\nu(|f|)$ for $f \in L$.

Note. If $\mathscr{L} = (L, \nu)$ is a real integrable space and $(Y, |\cdot|)$ is an abelian complete normed group, then I is an integral on $\mathscr{L}$ with values in Y (see Definition 2 and Proposition 1 of [13]).

PROPOSITION 7. *$\mathscr{L} = (L, \nu)$ is a real integrable space if and only if, for any sequence $\{f_n\} \subset L^+$ with*

$$\lim_{n\to\infty} \downarrow f_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \nu(f_n - f_{n+1}) < \infty,$$

we have

$$\lim_{n\to\infty} |I(f_n)| = 0.$$

Proof. Since

$$f_n = \sum_{k=n}^{\infty} (f_k - f_{k+1})$$

provided that

$$\lim_{n\to\infty} f_n = 0,$$

the necessity follows from the inequality $|I(f)| \leqslant \nu(f)$ for $f \in L^+$.

Conversely, let

$$f \leqslant \sum_{n=1}^{\infty} f_n \quad \text{and} \quad \sum_{n=1}^{\infty} \nu(f_n) < \infty \quad (f, f_n \in L^+).$$

Choose a $g \in L^+$ with $g \leqslant f$, and take a sequence $\{g_n\} \subset L^+$ (corresponding to g and f_n) as in Lemma 3. Since

$$\lim_{m\to\infty} \downarrow \Big(g - \sum_{n=1}^{m} g_n\Big) = 0 \quad \text{and} \quad \Big(g - \sum_{n=1}^{m} g_n\Big) - \Big(g - \sum_{n=1}^{m+1} g_n\Big) = g_{m+1} \leqslant f_{m+1},$$

the assumption implies

$$\lim_{m\to\infty} \Big| I\Big(g - \sum_{n=1}^{m} g_n\Big)\Big| = 0.$$

Hence

$$|I(g)| = \lim_{m\to\infty}\Big|I\Big(\sum_{n=1}^{m} g_n\Big)\Big| \leqslant \sum_{n=1}^{\infty}|I(g_n)| \leqslant \sum_{n=1}^{\infty}\nu(f_n).$$

It follows that

$$\nu(f) \leqslant \sum_{n=1}^{\infty}\nu(f_n).$$

COROLLARY 4. *$\mathscr{L} = (L, \nu)$ is a real integrable space having the Fatou property if*

$$\lim_{n\to\infty}\downarrow f_n = 0 \textit{ implies } \lim_{n\to\infty}|I(f_n)| = 0 \ (f_n \in L^+).$$

Proof. By Proposition 7, $\mathscr{L}$ is a real integrable space. Suppose that

$$f, f_n \in L^+ \quad \text{and} \quad f \leqslant \lim_{n\to\infty}\uparrow f_n.$$

Then we have

$$\lim_{n\to\infty}|I(f - f_n \wedge f)| = 0,$$

and so

$$|I(f)| = \lim_{n\to\infty}|I(f_n \wedge f)| \leqslant \lim_{n\to\infty}\uparrow \nu(f_n).$$

It follows that $\mathscr{L}$ has the Fatou property.

COROLLARY 5. *$\mathscr{L} = (L, \nu)$ is a real integrable space having the Daniel property if and only if*

$$g_n \leqslant f_n \textit{ and } \lim_{n\to\infty}\downarrow f_n = 0 \textit{ imply } \lim_{n\to\infty}|I(g_n)| = 0 \ (g_n, f_n \in L^+).$$

Remark 5. Suppose $\mathscr{L} = (L, \nu)$ is a real integrable space. Then $\mathscr{L}$ has the saturability property if and only if

$$\sum_{n=1}^{\infty} f_n \leqslant f \text{ implies } \lim_{n\to\infty}|I(f_n)| = 0 \ (f_n, f \in L^+).$$

The starting point for Pellaumail's generalization of the classical Daniell theory are the following definitions (see [10], p. 1228).

Let Y be an abelian Hausdorff topological group.

Definition 7. A pair $\mathscr{D} = (L, I)$ is a *Daniell space* if

(a) L is a Riesz subgroup of R^S,

(b) I is an additive mapping from L into Y such that

$$\lim_{n\to\infty}\downarrow f_n = 0 \text{ implies } \lim_{n\to\infty} I(f_n) = 0 \ (f_n \in L^+).$$

Definition 8. A Daniell space $\mathscr{D} = (L, I)$ has the *saturability property* if

(S′) $$\sum_{n=1}^{\infty} f_n \leqslant f \text{ implies } \lim_{n\to\infty} I(f_n) = 0 \ (f_n, f \in L^+).$$

LEMMA 4. *Let $\mathscr{D} = (L, I)$ be a Daniell space having the saturability property. There exists a group topology τ on L^σ with the following properties:*

(i) *The mapping φ: $L^\sigma \to L^\sigma$, defined by $\varphi(f) = f^+$ for $f \in L^\sigma$, is τ-continuous.*

(ii) *If $\{f_n\}$ is an order-bounded sequence in L^σ and $\lim_{n\to\infty} f_n = f$, then*

$$\tau\text{-}\lim_{n\to\infty} f_n = f.$$

(iii) *L is τ-dense in L^σ.*

(iv) *I is continuous with respect to the topology induced on L by τ.*

Proof. By Theorem 4, for any ν: $L^+ \to R^+$ such that (L, ν) is a real integrable space having the saturability property, we have $L^\sigma \subset \bar{L}$. Let τ be the lower upper bound of the family of those topologies on L^σ which are generated by all ν with the above property (see [13], p. 912). Clearly, τ is a group topology. Moreover, it is easy to see that (i) holds (cf. the proof of Proposition 3). Assertion (ii) follows directly from the implication (ii) $\Rightarrow$ (iv) of Theorem 4.

From (i), (ii) and Remark 1 we infer that the τ-closure of L is a σ-reticulated Riesz subgroup of R^S. Hence (iii) follows.

Finally, to establish the continuity of I, it is enough to show that, for each continuous bounded seminorm $|\cdot|$ on Y, the mapping $|I|$: $L \to R^+$ is continuous (see [6], p. 68 and 76). Let ν be associated with $|\cdot|$ according to the construction following Lemma 3. By Corollary 4 and Remark 5, (L, ν) is a real integrable space having the saturability property. Hence Remark 4 yields the desired continuity of $|I|$.

Definition 9. A Daniell space $\mathscr{D} = (L, I)$ has the *Lebesgue property* if

(L′) $\quad |f_n| \leqslant g$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} f_n = f$ imply

$$f \in L \text{ and } \lim_{n\to\infty} I(f_n) = I(f) \quad (f_n, g \in L).$$

THEOREM 6 ([10], p. 1234). *Let Y be a sequentially complete abelian Hausdorff topological group. A Daniell space $\mathscr{D} = (L, I)$, where I: $L \to Y$, has the saturability property if and only if there exists a mapping I^σ: $L^\sigma \to Y$ such that $I^\sigma | L = I$ and $\mathscr{D}^\sigma = (L^\sigma, I^\sigma)$ is a Daniell space having the Lebesgue property. The mapping I^σ satisfying the listed conditions, if exists, is unique.*

Proof. The sufficiency is obvious.

Necessity. Denote by $\tilde{Y}$ the completion of Y. By Lemma 4 (iii) and (iv), I can be uniquely extended to a τ-continuous additive mapping I^σ: $L^\sigma \to \tilde{Y}$ (cf. [3], Chapter 3, Section 3, No. 3 and 5). Lemma 4 (ii) shows that $\mathscr{D}^\sigma = (L^\sigma, I^\sigma)$ has the Lebesgue property. Hence it remains to prove that $I^\sigma(L^\sigma) \subset Y$. Put $G = \{f \in L^\sigma\colon I^\sigma(f) \in Y\}$. Clearly, $L \subset G$.

Since, for any sequence $\{f_n\} \subset L^\sigma$ such that

$$f_n \leqslant g \in L^\sigma \ (L^\sigma \ni g \leqslant f_n) \quad \text{and} \quad \lim_{n\to\infty} \uparrow f_n = f \ \ (\lim_{n\to\infty} \downarrow f_n = f),$$

$\{I^\sigma(f_n)\}$ is a τ-Cauchy sequence, G is σ-reticulated. By Lemma 2, it now follows that $G = L^\sigma$.

A similar reasoning also establishes the uniqueness of I^σ.

Let us note that an analogue of Theorem 5 for Daniell spaces holds true. The proof is essentially the same.

THEOREM 7. *Let $\mathscr{D}_0 = (L_0, I_0)$ be a Daniell space having the Lebesgue property. If $\mathscr{D} = (L, I)$ is a Daniell space such that $L \subset L_0$ and $I = I_0|L$, then $L^\sigma \subset L_0$ and $I^\sigma = I_0|L^\sigma$, where $\mathscr{D}^\sigma = (L^\sigma, I^\sigma)$ is as in Theorem 6.*

5. Beppo Levi and additivity properties. In this section we study two properties of real integrable spaces which are stronger than the saturability property.

Definition 10 (cf. [1], p. 69). A real integrable space $\mathscr{L}$ has the *Beppo Levi property* if

$$\text{(BL)} \qquad \sup_m \nu\Big(\sum_{n=1}^{m} f_n\Big) < \infty \text{ implies } \lim_{n\to\infty} \nu(f_n) = 0 \ \ (f_n \in L^+).$$

The Beppo Levi property resembles the so-called Axiom A introduced for Banach spaces by Gould [5], p. 686 (cf. also condition (0) in [9], p. 801, or in [11], p. 106). Let us also note that, under the name "semiadditivity", a similar property has been considered by Schäfke [12], p. 160 and ff. Our terminology is justified by the forthcoming Proposition 8.

Obviously, if $\mathscr{L}$ has the Beppo Levi property, then it also has the saturability property. The converse fails to be true in general.

Example 4. Suppose (S, Σ, μ) is a (positive) complete finite measure space. Let L be the family of all measurable real-valued functions on S, and put

$$\nu(f) = \int_S f \wedge 1 \, d\mu \quad \text{for } f \in L^+.$$

Clearly, (L, ν) is a real integrable space having the Lebesgue property, but it has not the Beppo Levi property unless $\mu(S) = 0$.

The proof of the following proposition is analogous to that of Proposition 5:

PROPOSITION 8. *$\mathscr{L}$ has the Beppo Levi property if and only if every monotone sequence $\{f_n\} \subset L$ bounded in the seminorm N is a Cauchy sequence in L^*.*

THEOREM 8. *If $\mathscr{L}$ has the Beppo Levi property, then $\bar{\mathscr{L}}$ also has this property.*

Proof. Suppose that

$$f_n \in \bar{L}^+ \quad \text{and} \quad \sup_m \bar{\nu}\Big(\sum_{n=1}^{m} f_n\Big) < \infty .$$

By Proposition 3, there are $g_n \in L^+$ with $N(f_n - g_n) < 2^{-n}$. Since

$$\sup_m \nu\Big(\sum_{n=1}^{m} g_n\Big) \leqslant \sup_m \bar{\nu}\Big(\sum_{n=1}^{m} f_n\Big) + \sum_{n=1}^{\infty} N(f_n - g_n) < \infty,$$

we get

$$\lim_{n\to\infty} \nu(g_n) = 0 .$$

Hence

$$\lim_{n\to\infty} \bar{\nu}(f_n) = 0 .$$

THEOREM 9. *Let $\mathscr{L}$ be a real integrable space having the Beppo Levi property. If*

$$\{f_n\} \subset \bar{L}, \quad \sup_m N\Big(\bigvee_{n=1}^{m} |f_n|\Big) < \infty \quad \text{and} \quad \lim_{n\to\infty} f_n = f \in R^S,$$

then

$$f \in \bar{L} \quad \text{and} \quad \lim_{n\to\infty} N(f_n - f) = 0 .$$

Proof. Put

$$g = \lim_{m\to\infty} \bigvee_{n=1}^{m} |f_n| .$$

Theorem 8 and Propositions 8 and 1 yield $g \in \bar{L}$. Now the implication (ii) $\Rightarrow$ (iv) of Theorem 4 gives the desired result.

Definition 11. A real integrable space $\mathscr{L}$ has the *additivity property* if

$$\text{(A)} \qquad \nu(f+g) = \nu(f) + \nu(g) \quad (f, g \in L^+).$$

It follows from Proposition 3 that if $\mathscr{L}$ has the additivity property, then $\bar{\mathscr{L}}$ also has this property. Clearly, $\mathscr{L}$ has the Beppo Levi property provided that it has the additivity property. The converse need not hold.

Example 5. Suppose (S, Σ, μ) is a measure space. Put $L = L^p(\mu)$, where $1 < p < \infty$, and

$$\nu(f) = \Big(\int_S f^p \, d\mu\Big)^{1/p} \quad \text{for } f \in L^+ .$$

(L, ν) is a real integrable space having the Beppo Levi property. (This follows from Proposition 14 and Theorem 2 of [4], Section 12, p. 224, and Proposition 8 above; cf. also Theorem 3.1 of [5].) It has not the additivity property unless there are no disjoint sets $A, B \in \Sigma$ with $0 < \mu(A)$, $\mu(B) < \infty$.

The author is indebted to Dr. Z. Lipecki for many valuable suggestions and improvements.

Added in proof. Some interesting results related to the subject of the present paper (especially of Section 4) can be found in a recent work by N. J. Kalton, *Topologies on Riesz groups and applications to measure theory*, Proceedings of the London Mathematical Society (3) 28 (1974), p. 253-273.

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Reçu par la Rédaction le 25. 2. 1974