COLLOQUIUM MATHEMATICUM

VOL. XLV 1981 FASC. 2

THE FIXED POINT PROPERTY FOR SET-VALUED MAPPINGS

BY

T. MAĆKOWIAK (WROCŁAW)

In this paper we introduce componentwise continuous (c.c.) multifunctions (see the definition below) and we use these functions to obtain some fixed point theorems which generalize most known fixed point theorems for trees, dendroids, and λ-dendroids. In particular, we obtain the following characterization of trees: a Hausdorff continuum X is a tree if and only if every c.c. function from X into itself has a fixed point. C.c. functions need not have the fixed point property for dendroids, but they have an almost fixed point (as have refluent mappings). Fans and smooth dendroids have the fixed point property for c.c. closed mappings. If X is a λ-dendroid, we show only that point connected c.c. functions on X have an almost fixed point (for all c.c. functions it is an open question).

These results imply those of Wallace, Ward, Mańka, En-Nashef, Smithson, Muenzenberger, and of some others; they imply the fixed point property for λ-dendroids and closed mappings which preserve continua (in the metric case). We conclude also the existence of a fixed point different from a given end-point (not belonging to a given end-continuum for λ-dendroids). Some results on arclike continua and an almost fixed point property for c.c. mappings are also obtained.

All spaces considered in the paper are Hausdorff compact. The results used to manipulate nets and nets of sets can be found in [3], [5], and [15].

1. Preliminaries. A multifunction F: $X \to Y$ denotes a point-to-set correspondence such that $F(x)$ is a nonempty subset of Y for each $x \in X$. A multifunction $F: X \to Y$ is said to be

(i) *point closed* if $F(x)$ is closed for each $x \in X$;

(ii) *point connected* if $F(x)$ is connected for each $x \in X$;

(iii) *closed* if $F(A)$ is closed for each A closed in X;

(iv) *continuum-valued* if $\overline{F(A)}$ is a continuum for each continuum $K \subset X$;

(v) *refluent* if for each subcontinuum K of X and for each component C of $\overline{F(K)}$ we have $F(x) \cap C \neq \emptyset$ for each $x \in K$ (see [4], p. 524);

(vi) *lower semi-continuous* (l.s.c.) if $F^{-1}(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$ is open for each open $B \subset Y$;

(vii) *upper semi-continuous* (u.s.c.) if $F^{-1}(B)$ is closed for each closed $B \subset Y$;

(viii) *continuous* if F is both l.s.c. and u.s.c.

We say that $F\colon X \to Y$ is *componentwise continuous* (c.c.) if $x = \lim\{x_\sigma\}$ implies that

(A) $\mathrm{Ls}\{C_\sigma\} \cap F(x) \neq \emptyset$, where C_σ is a component of $F(x_\sigma)$ for each σ;

(B) every component of $F(x)$ intersects $\mathrm{Ls}\{F(x_\sigma)\}$.

Using the fact that every u.s.c. and point closed multifunction is closed (see [14], 9.6, p. 180) and making use of Theorems 1 and 2 in [7], p. 61-62 (one can extend these theorems to the nonmetric case), it is easy to show that

PROPOSITION 1.1. *The following implications hold for multifunctions and none can be reversed*:

$$\begin{tikzcd}
 & \begin{array}{l}\textit{l.s.c.,}\\ \textit{point connected,}\\ \textit{closed}\end{array} \arrow[r] \arrow[ddr] & \begin{array}{l}\textit{l.s.c.,}\\ \textit{point}\\ \textit{connected}\end{array} \arrow[dr] & & \\
 & & & \begin{array}{l}\textit{c.c.,}\\ \textit{point con-}\\ \textit{nected}\end{array} \arrow[dr] & \\
\begin{array}{l}\textit{single-}\\ \textit{-valued,}\\ \textit{continuous}\end{array} \arrow[uur] \arrow[r] \arrow[ddr] & \begin{array}{l}\textit{u.s.c.,}\\ \textit{point connected,}\\ \textit{point closed}\end{array} \arrow[r] \arrow[ddr] & \begin{array}{l}\textit{c.c., point}\\ \textit{connected,}\\ \textit{closed}\end{array} \arrow[ur] \arrow[dr] & & \textit{c.c.} \arrow[dd] \\
 & & & \begin{array}{l}\textit{continuum-}\\ \textit{-valued}\end{array} \arrow[dr] & \\
 & \begin{array}{l}\textit{continuous,}\\ \textit{point closed}\end{array} \arrow[r] & \begin{array}{l}\textit{refluent,}\\ \textit{u.s.c.,}\\ \textit{point closed}\end{array} \arrow[rr] & & \textit{refluent}
\end{tikzcd}$$

One can also show that every point closed, u.s.c. and refluent multifunction from a locally connected space is c.c. Some other properties of c.c. multifunctions can be established. In particular, the composition of two closed c.c. multifunctions is c.c., and the closure of c.c. multifunctions is also c.c. In what follows we use only the following

PROPOSITION 1.2. *Let $F\colon X \to Y$ be a c.c. multifunction and let $f\colon Y \to Z$ be a single-valued continuous function. Then $f \circ F$ is c.c.*

In fact, let $x = \lim\{x_\sigma : \sigma \in \Sigma\}$, where $x \in X$ and $x_\sigma \in X$ for σ belonging to an arbitrary directed set Σ, let C_σ be a component of $f \circ F(x_\sigma)$ for $\sigma \in \Sigma$, and let $y_\sigma \in F(x_\sigma)$ be such that $f(y_\sigma) \in C_\sigma$. For $\sigma \in \Sigma$ take a component K_σ of $F(x_\sigma)$ such that $y_\sigma \in K_\sigma$. Since the function F is c.c., there is a point $y_0 \in \mathrm{Ls}\{K_\sigma\colon \sigma \in \Sigma\} \cap F(x)$. Let U be an open neighbourhood of y_0 and

$$\Sigma_U = \{\sigma \in \Sigma\colon K_\sigma \cap U \neq \emptyset\}.$$

For $\sigma \in \Sigma_U$ take $y_{\sigma,U} \in K_\sigma \cap U$. Let the set

$$\Pi = \{(\sigma, U): \sigma \in \Sigma_U, U \text{ is a neighbourhood of } y_0\}$$

be directed by the relation $\leqslant$ such that $(\sigma, U) \leqslant (\sigma', U')$ if and only if $U' \subset U$ and $\sigma \leqslant \sigma'$. Then

$$y_0 = \lim\{y_{\sigma,U}: (\sigma, U) \in \Pi\}.$$

Since f is continuous, it follows that

$$f(y_0) = \lim\{f(y_{\sigma,U}): (\sigma, U) \in \Pi\} \subset \mathrm{Ls}\{f(K_\sigma): \sigma \in \Sigma\} \subset \mathrm{Ls}\{C_\sigma: \sigma \in \Sigma\}$$

and $f(y_0) \in f \circ F(x)$. Therefore, condition (A) of the definition of c.c. multifunction is satisfied.

Now, let C be an arbitrary component of $f \circ F(x)$ and let K be a component of $F(x)$ such that $f(K) \subset C$. Since F is c.c., we have

$$\mathrm{Ls}\{F(x_\sigma): \sigma \in \Sigma\} \cap C \neq \emptyset.$$

Similarly as above, if $y_0 \in \mathrm{Ls}\{F(x_\sigma): \sigma \in \Sigma\} \cap C$, then

$$f(y_0) \in \mathrm{Ls}\{f \circ F(x_\sigma): \sigma \in \Sigma\} \cap f(C).$$

Therefore $f \circ F$ is c.c.

Let $F: X \to Y$ and put, for each $x \in X$,

$$F^*(x) = \bigcap\{\overline{F(U)}: U \text{ is a neighbourhood of } x\},$$

$$F_*(x) = \bigcap\{\overline{F(C)}: C \text{ is a component of } x \text{ in a neighbourhood of } x\}.$$

It is easy to show that F^* is a point closed u.s.c. multifunction and that if F satisfies condition (A) of the definition of c.c. multifunction, then $\overline{F(x)}$ intersects each component of $F^*(x)$ for $x \in X$. Therefore,

PROPOSITION 1.3. *If F is c.c., then F^* is c.c., u.s.c., and point closed. Moreover, if F is point connected c.c., then F^* is continuum-valued.*

Similarly, one can show that if X' is a locally connected subcontinuum of X, then $F_*|X'$ is u.s.c. and point closed. Furthermore

PROPOSITION 1.4. *If $F: X \to Y$ is refluent and X' is a hereditarily locally connected subcontinuum of X, then $F_*|X'$ is refluent, closed, and u.s.c.*

It remains to prove, by 9.6 in [14], p. 180, that $F_*|X'$ is refluent. Let K be a subcontinuum of X' and let Q be a component of $F_*(K)$. Then $Q = \bigcap\{V_a\}$, where $\{V_a\}$ is the family of open and closed sets in $F_*(K)$ containing Q. Fix V_a and consider the set

$$A = \{x \in K: F_*(x) \cap V_a \neq \emptyset\}.$$

Since V_a is closed in $F_*(K)$ and $F_*(K)$ is closed in Y, we infer that A is closed in K by the upper semi-continuity of $F_*|X'$. Now, by the normality of Y, there are open disjoint sets G and H in Y such that $V_a \subset G$ and $F_*(K)\setminus V_a \subset H$. Let $x_0 \in A$. There is a neighbourhood U of x_0 in X such that if C is a component containing x_0 in U, then $\overline{F(C)} \subset G \cup H$. Moreover, $\overline{F(C)} \cap G \neq \emptyset$ because $F_*(x_0) \subset \overline{F(C)}$. Therefore, there is a component R of $\overline{F(C)}$ which is contained in G. Thus, if C' is a component of $C \cap K$ containing x_0, then $F(x) \cap G \neq \emptyset$ for $x \in C'$ by the refluence of F. Consequently, $F_*(x) \cap \bar{G} \neq \emptyset$ for $x \in C'$. Since $\bar{G} \cap F_*(K) = V_a$, we have $F_*(x) \cap V_a \neq \emptyset$ for $x \in C'$. But K is locally connected, and so C' is a neighbourhood of x_0 in K which is contained in A. Hence A is also open, which implies $A = K$. Therefore F_* is refluent.

Recall that $F\colon X \to X$ has the fixed point x if $x \in F(x)$. We say that $x \in X$ is an *almost fixed point* of F if for each neighbourhood U of x in X we have $U \cap F(U) \neq \emptyset$.

For every point a and for every subcontinuum K of hereditarily unicoherent continuum X there is a unique minimal continuum in X which intersects sets $\{a\}$ and K; we denote it by aK. In particular, ab denotes the unique continuum irreducible between a and b. We use also the following notation: $I(a, b) = \{x \in X\colon xb = ab\}$; $x \leqslant_p y$ means $px \subset py$; and $M_p(x) = \{y \in X\colon x \leqslant_p y\}$.

In the next sections we consider the following classes of hereditarily unicoherent continua: arcs, trees, fans, dendroids, smooth dendroids, λ-dendroids, arclike continua, and treelike continua. Recall that *λ-dendroid* (*dendroid*, *tree*) is a hereditarily unicoherent and hereditarily decomposable (arcwise connected, locally connected) Hausdorff continuum. A (*generalized*) *arc* is a continuum which has exactly two nonseparating points. If a point p of a dendroid X is the common end-point of three (or more) arcs in X whose only common point is p, then p is called a *ramification point*. A dendroid X having exactly one ramification point p is called a *fan* and p is called a *top* of X. A dendroid X is said to be *smooth at p* if the order $\leqslant_p$ is closed. We say that X is *smooth* if there is $p \in X$ such that X is smooth at p.

A metric continuum X is said to be *arclike* (*treelike*) if for each $\varepsilon > 0$ there exists a (single-valued) continuous function f from X onto $[0, 1]$ (onto a metric tree D) such that $\operatorname{diam} f^{-1}(t) < \varepsilon$ for $t \in [0, 1]$ (for $t \in D$, respectively).

2. Fixed point theorems. First we deduce some consequences from the assumption that continuum has the fixed point property for some class of multifunctions.

THEOREM 2.1. *If a continuum X is not hereditarily unicoherent, then there is a point connected, closed u.s.c. (l.s.c.) multifunction F from X into itself which is fixed point free.*

Proof. In the Cartesian coordinates in the plane we put $D = \{(x_1, x_2): x_1^2+x_2^2 \leqslant 1\}$ and

$$S_1 = \{(x_1, x_2):\ x_1^2+x_2^2 = 1 \text{ and } x_2 \geqslant 0\},$$

$$S_2 = \{(x_1, x_2):\ x_1^2+x_2^2 = 1 \text{ and } x_2 \leqslant 0\}, \quad \text{and} \quad S = S_1 \cup S_2.$$

If the continuum X is not hereditarily unicoherent, then there are continua M_1 and M_2 such that $M_1 \cap M_2 = A_1 \cup A_2$, where sets A_1 and A_2 are closed, nonempty, and disjoint. By Tietze's theorem there is a continuous single-valued mapping f from X into D such that $f(A_1) = (-1,\ 0)$, $f(A_2) = (1,\ 0)$, $f(M_1) = S_2$, and $f(M_2) = S_1$. Moreover, there is a continuous point closed and point connected multifunction G from D onto S such that $G(x) = x$ for $x \in S$ (see [8], p. 423).

Now, let U_1 and U_2 be open and disjoint subsets of D such that $(1,\ 0) \in U_1$ and $(-1,\ 0) \in U_2$, and let a_i be an arbitrary point of A_i for $i = 1, 2$. We define H: $S \to M_1 \cup M_2$ as follows:

$$H(x) = \begin{cases} M_i & \text{if } x \in S_i \setminus (U_1 \cup U_2), \\ a_i & \text{if } x \in U_i. \end{cases}$$

Taking $F(x) = H\big(G(f(x))\big)$ for $x \in X$ we obtain a point connected, closed and u.s.c. multifunction from X into X. It is fixed point free because $F(X) \subset M_1 \cup M_2$, $F(A_1) = a_2$, $F(A_2) = a_1$, $F(M_1) \subset M_2$, and $F(M_2) \subset M_1$.

If V_1 and V_2 are open subsets of X such that $A_1 \subset V_1$, $A_2 \subset V_2$, and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, then a multifunction F': $X \to X$ defined by

$$F'(x) = \begin{cases} M_1 \cup M_2 & \text{if } x \in X \setminus (M_1 \cup M_2), \\ M_i & \text{if } x \in M_j \setminus (\overline{V}_1 \cup \overline{V}_2) \text{ and } i \neq j, \\ a_i & \text{if } x \in \overline{V}_j \text{ and } i \neq j \end{cases}$$

is a point connected, closed and l.s.c. multifunction from X into itself which does not have a fixed point. The proof of Theorem 2.1 is complete.

If in Theorem 2.1 we assume additionally that X is metric, then we can construct F which is continuous (cf. [8], Theorem 2, p. 422). Such an implication is not true for Hausdorff continua. We have

Example 2.1. The closed ordinal space $[0,\ \Omega]$ consists of the set of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. We obtain a Hausdorff arc A from $[0,\ \Omega]$ by placing between each ordinal α and its successor $\alpha+1$ a copy of the unit interval $I = (0,\ 1)$ and we give A the order topology. We obtain a space B from A by φ which identifies 0 with Ω. Then B is a Hausdorff continuum (circle) which is not unicoherent. Moreover, if F: $B \to B$ is a continuous point closed multifunction, then F has a fixed point.

In fact, let F be fixed point free. Then, in particular, $\varphi(0) \notin F\big(\varphi(0)\big)$. Consider the set R of all points x of B such that $F(x) \subset B \setminus \{\varphi(0)\}$

and let C be a component of $\varphi^{-1}(R)$ containing 0. Since F is continuous, we have $y \leqslant \varphi^{-1}\big(F(\varphi(y))\big)$ for each $y \in C$. Let y_0 be the supremum of C. Then $y_0 \neq \Omega$. Thus there is a countable sequence $\{y_n\}$ of points of C such that $\lim\{y_n\} = y_0$. For $n = 1, 2, \ldots$ the supremum of the set $\varphi^{-1}\big(F(\varphi(y_n))\big)$ is denoted by z_n. From the continuity of F it follows that $\lim\{z_n\} \in \varphi^{-1}\big(F(\varphi(y_0))\big)$. The choice of y_0 and y_n implies $\Omega = \lim\{z_n\}$ and $z_n \neq \Omega$, which is impossible because A does not have a countable local basis at Ω.

THEOREM 2.2. *If a continuum X has the fixed point property for c.c. point connected multifunctions, then X is a tree.*

Proof. First we have

(1) *Every nondegenerate subcontinuum K of X with the empty interior separates X.*

In fact, suppose that the set $X \setminus K$ is connected and let a and b be distinct points of K. It is easy to check that the multifunction $F\colon X \to X$ defined by

$$F(x) = \begin{cases} K & \text{if } x \in X\setminus K, \\ (X\setminus K)\cup\{a\} & \text{if } x \in K\setminus\{a\}, \\ (X\setminus K)\cup\{b\} & \text{if } x = a \end{cases}$$

is point connected c.c. and F has no fixed point, a contradiction.

(2) *If K is a subcontinuum of X, then K has also the fixed point property for c.c. point connected multifunctions.*

Indeed, let $F\colon K \to K$ be c.c. and point connected. Then

$$G(x) = \begin{cases} K & \text{if } x \in X\setminus K, \\ F(x) & \text{if } x \in K \end{cases}$$

is a c.c. point connected multifunction. If $x \in X$ is a fixed point of G, then it is a fixed point of F.

(3) *X is hereditarily decomposable.*

It follows from (1) that X is decomposable. Similarly, every subcontinuum of X is decomposable by (2).

(4) *If K is a subcontinuum of X, then K is arcwise connected.*

In fact, let M be a subcontinuum of K which is irreducible between x and y. It follows from (3) and from Theorem 2.7 in [6], p. 650, that there is a decomposition $\mathscr{D}$ such that $\mathscr{D}$ is u.s.c., $M/\mathscr{D}$ is a generalized arc, and each element of $\mathscr{D}$ has an empty interior. Combining this conclusion with (1) and (2) we infer that $\mathscr{D}$ has degenerate elements, thus M is a generalized arc, i.e. condition (4) holds.

Now we will prove that

(5) *X is hereditarily locally connected.*

Suppose, on the contrary, that X is not hereditarily locally connected. Then there is a nondegenerate subcontinuum K_0 of X and there is a net $\{K_\sigma\colon \sigma \in \Sigma\}$ of subcontinua of X converging to K_0 such that for all σ and σ' in Σ either $K_\sigma = K_{\sigma'}$ or $K_\sigma \cap K_{\sigma'} = \emptyset$ and $K_\sigma \cap K_0 = \emptyset$ (see [20], Theorem 3; cf. [7], Theorem 2, p. 269, for the metric case).

Now, let a and b denote two distinct points in K_0 and let ab be an arc in K_0 (cf. (4)) having a and b as its nonseparating points. For each $\sigma \in \Sigma$ take an arc $p_\sigma q_\sigma$ in X having p_σ and q_σ as nonseparating points and such that $p_\sigma q_\sigma \cap ab = \{p_\sigma\}$, $p_\sigma q_\sigma \cap K_\sigma = \{q_\sigma\}$. We can assume (see [3], (3.1.23), p. 172) that a net $\{p_\sigma\colon \sigma \in \Sigma\}$ is convergent to some point p and that there is a nondegenerate arc cd in ab such that $c < d < d' < p$ (in the order of ab) for some d' and $p_\sigma \in d'p$ for each $\sigma \in \Sigma$. Put

$$K = \bigcup_{\sigma\in\Sigma}(K_\sigma \cup p_\sigma q_\sigma) \cup d'p.$$

Then the set K is connected. The multifunction $F\colon X \to X$ defined by

$$F(x) = \begin{cases} cd & \text{if } x \in X\setminus cd, \\ K \cup \{c\} & \text{if } x \in cd\setminus\{c\}, \\ K \cup \{d\} & \text{if } x = c \end{cases}$$

is point connected and c.c. Moreover, F has no fixed point, a contradiction.

It follows from Proposition 1.1 and Theorem 2.1 that X is hereditarily unicoherent. Therefore X is a tree by (5). The proof of Theorem 2.2 is complete.

Some characterization of trees follows from Theorem 2.2. To formulate it we need the following

THEOREM 2.3. *Let X be a tree, $p \in X$, and let a multifunction $F\colon X \to X$ be c.c. If $x_0 \in X$ is such that $x_0 \leqslant_p y_0$ for some $y_0 \in F(x_0)$, then there exists an $x \in X$ such that $x_0 \leqslant_p x$ and $x \in F(x)$.*

Proof. The set $R = \{x \in X\colon x_0 \leqslant_p x \text{ and } F(x) \cap M_p(x) \neq \emptyset\}$ is nonempty because $x_0 \in R$. Moreover, if there is no fixed point x of F such that $x_0 \leqslant_p x$, then R is closed.

Indeed, let $x = \lim\{x_\sigma\}$ and $x_\sigma \in R$ for each σ. Then $\mathrm{Ls}\{M_p(x_\sigma)\} \subset M_p(x)$ by Lemma (2.8) in [16], p. 96. Let $\{C_\sigma\}$ be a net of components of $F(x_\sigma)$ with $C_\sigma \cap M_p(x_\sigma) \neq \emptyset$. Then $C_\sigma \subset M_p(x_\sigma)$ for each σ by Lemma 4 in [21], p. 352. Therefore $\mathrm{Ls}\{C_\sigma\} \subset M_p(x)$. Furthermore, $\mathrm{Ls}\{C_\sigma\} \cap F(x) \neq \emptyset$ by condition (A) of the definition of c.c. multifunction. Thus $F(x) \cap M_p(x) \neq \emptyset$, which means that $x \in R$.

Now, let a be a maximum in R (see [24], Theorem 1). Then we find $b \in F(a) \cap M_p(a)$. Consider a net $\{a_\sigma\}$ such that $a = \lim\{a_\sigma\}$ and $a <_p a_\sigma <_p b$ for each σ. Denote by C a component of $F(a)$ containing b. Since F is c.c. (condition (B)), there is a $c \in C \cap \mathrm{Ls}\{F(a_\sigma)\}$. Points b and c belong to C, thus $ab \cap ac = ad$ for some $d \neq a$ (otherwise, $a \in C \subset F(a)$). We can assume that $a_\sigma \in ad$ for each σ. Since the set $M_p(a) \setminus \{a\}$ is open (see [25]), there is an open connected subset U of X such that $c \in U \subset M_p(a)$ and $U \cap aa_0 = \emptyset$ for some a_0 such that $a <_p a_0 \leqslant_p d$. There is σ such that $F(a_\sigma) \cap U \neq \emptyset$ and $a_\sigma \in aa_0$. The construction implies $a <_p a_\sigma$ and $F(a_\sigma) \cap M_p(a_\sigma) \neq \emptyset$. Therefore $a_\sigma \in R$. Consequently, the point a is not a maximum of R, a contradiction.

Applying Theorem 2.2 and putting $x_0 = p$ in Theorem 2.3 we obtain

COROLLARY 2.1. *Let X be a continuum. Then the following conditions are equivalent*:

(i) *the continuum X is a tree*;

(ii) *X has the fixed point property for c.c. multifunctions*;

(iii) *X has the fixed point property for c.c. point connected multifunctions.*

Corollary 2.1 implies (cf. Proposition 1.1) Theorem 1 from [22] and Theorem A from [23]. As can be seen from the following example, a dendroid need not have the fixed point property even for c.c. closed multifunctions.

Example 2.2. Let (x, y) denote a point of the Euclidean plane R^2 having x and y as its rectangular coordinates and let $A(p, q)$ stand for the straight-line segment joining p and q. Put

$$B = A\big((0,0),(1,0)\big), \qquad C = A\big((1,1),(1,0)\big),$$
$$A_n = A\Big(\big(0, 1/(n+1)\big), (1, 1/n)\Big) \quad \text{for } n = 1, 2, \ldots$$

and let

$$X = B \cup C \cup \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad Y = X \cup \varphi(X),$$

where $\varphi(x, y) = (1-x, -y)$ for $(x, y) \in R^2$. For each real t consider

$$K(t) = \{(x, y)\colon t \leqslant |y|\}$$

and

$$L(t) = \{(x, 0)\colon x \leqslant t^2 \text{ or } \sqrt{|t|} \leqslant x\} \cup C \cup \varphi(C)$$

and define multifunctions $F\colon X \to X$ and $G\colon Y \to Y$ by

$$F(x, y) = \begin{cases} B \cup C \cup \bigcup\limits_{m \neq n} A_m & \text{if } (x, y) \in A_n \setminus C, \\ (0, 0) & \text{if } (x, y) \in C, \\ C \cup \big(K(1-x) \cap X\big) & \text{if } (x, y) \in B \setminus \{(1, 0)\} \end{cases}$$

and

$$G(x, y) = \begin{cases} B & \text{if } (x, y) \in Y \setminus B, \\ \varphi(x, y) & \text{if } (x, y) = (0, 0) \text{ or } (1, 0), \\ Y \cap \Big(L(x) \cup K\big(4x(1-x)\big)\Big) & \text{if } (x, y) \in B \setminus \{(0, 0), (1, 0)\}. \end{cases}$$

The space X is the so-called "harmonic brush". It is easy to verify that F is point closed, l.s.c., and continuum-valued, and that F has no fixed point. Similarly, it is easy to check that Y is a dendroid and that G is a c.c. closed multifunction without a fixed point.

It follows from Corollary 11 in [1], p. 308, and from Ward's lemma (see [27], p. 924) that

PROPOSITION 2.1. *Every increasing net $\{x_\sigma\}$ (in the order $\leqslant_p$) in a dendroid X is convergent, and if $x = \lim\{x_\sigma\}$, then $x_\sigma \leqslant_p x$ for each σ.*

We have the following result for dendroids:

THEOREM 2.4. *Let a multifunction $F: X \to X$ from a dendroid X into itself be such that $F(A)$ is closed and $F|A$ is refluent for each arc $A \subset X$. If $p \notin F(p)$ and $q \in F(p)$, then there is a maximal point x (in the order $\leqslant_p$) such that $px \cap pq \neq \{p\}$ and x is a limit of an increasing net $\{x_\sigma\}$ such that $x \in F(x_\sigma)$ for each σ.*

Proof. For $x \in X$ put $H(x) = F(x) \cup \bigcap \{F(x'x): x' <_p x\}$. Then $H(x)$ is compact for each $x \in X$ and (cf. Proposition 2.1) we have

(6) *If $\{x_\sigma\}$ is an increasing net and $x_0 = \lim\{x_\sigma\}$, then* $\operatorname{Ls}\{H(x_\sigma)\} \subset H(x_0)$.

Put $J = \{x \in X$: there exists a minimal point $t_x \in H(x)$ such that $x \leqslant_p t_x$ and $H(y) \cap yt_x = \emptyset$ for each $x <_p y <_p t_x\}$. We will prove

(7) *If $y_0 \in pz_0$ and $z_0 \in H(y_0)$, then there is an $x_0 \in J$ such that $y_0 \leqslant_p x_0$.*

Indeed, let K denote the set of all $x \in y_0 z_0$ such that $H(x) \cap xz_0 \neq \emptyset$. The set K is nonempty because $y_0 \in K$. Let $x_0 = \sup K$. Then $x_0 \in K \cap J$, i.e. (7) holds.

Let $\mathscr{A}$ be the family of all (in general transfinite) nets $\{x_a\}$ in J satisfying the condition $t_{x_a} \leqslant_p x_{a+1}$ for each a in its domain. It follows that if $\{x_a\}$ is in $\mathscr{A}$, then $x_1 <_p x_2 <_p \ldots <_p x_a <_p \ldots$ We order $\mathscr{A}$ partially in the following manner. If both $\{x_a\}$ and $\{y_\beta\}$ are elements of $\mathscr{A}$, then $\{x_a\}$ preceds $\{y_\beta\}$ provided $\{x_a\}$ is an initial segment of $\{y_\beta\}$. Clearly, if $\mathscr{C}$ is a chain in $\mathscr{A}$, then the union of $\mathscr{C}$ is again in $\mathscr{A}$. Thus, by Zorn's lemma, $\mathscr{A}$ contains a maximal element. Let $\mathscr{C} = \{x_a\}$ be a maximal element in $\mathscr{A}$ and $a = \sup\{x_a\}$. Then

(8) *There is a point y_0 such that $t_{x_a} \leqslant_p y_0$ for each a and $y_0 \in H(y_0)$.*

If $a \notin \mathscr{C}$, then $a = t_0$, where $t_0 = \lim\{t_{x_a}\}$. But $t_{x_a} \in H(x_a)$, so $t_0 \in H(a)$ by (6). If $a \in \mathscr{C}$ and $a <_p t_a$, then there is an increasing net $\{z_\sigma\}$ such that $a = \lim\{z_\sigma\}$ and $t_a \in \operatorname{Ls}\{F(z_\sigma)\}$ by the definition of H. Since $t_a \in H(a)$, we can assume that $t_a \in F(z_\sigma)$ for each σ. For each σ let C_σ be a component of $F(z_\sigma t_a)$ containing t_a and let

$$C = \bigcap_\sigma C_\sigma.$$

F is refluent, thus $F(t_a)\cap C_\sigma \neq \emptyset$ for each σ. Hence $F(t_a)\cap C \neq \emptyset$. Let z be a point of this intersection. Then $zt_a \subset C$ because C is a continuum. If $zt_a \cap at_a = \{t_a\}$, then we find $y_0 \in J$ such that $t_a \leqslant_p y_0$, which contradicts the maximality of $\mathscr{C}$ (because $\mathscr{C}\cup\{y_0\} \in \mathscr{A}$). Therefore $zt_a\cap at_a = z_1 t_a$ and $z_1 \neq t_a$. So let us construct an increasing net $\{a_\eta\}$ in $z_1 t_a$ such that $t_a = \lim\{a_\eta\}$ and $a_\eta <_p t_a$. We have $a_\eta \in z_1 t_a \subset zt_a \subset C \subset C_\sigma$ for each σ and, consequently, $a_\eta \in H(at_a)$. Let $b_\eta \in at_a$ be such that $a_\eta \in H(b_\eta)$. Since for each y which satisfies $a <_p y \leqslant_p t_a$ we have $H(y)\cap yt_a = \emptyset$, we conclude that $a_\eta <_p b_\eta <_p t_a$ for each η. Therefore $\lim\{a_\eta\} = \lim\{b_\eta\} = t_a$, thus $t_a \in H(t_a)$, which follows from $a_\eta \in H(b_\eta)$ and (6).

From (6)-(8) we infer that there is a point x such that $px\cap pq \neq \{p\}$, $x \in H(x)$, and x is maximal with respect to these properties, which is equivalent to the thesis of Theorem 2.4 by the definition of the mapping H.

Theorem 2.4 is related to the results from [17]. From Proposition 1.1 and Theorem 2.4 we obtain easily Ward's and En-Nashef's results (see [4], Theorem 3.5, p. 527; [27], Theorem 2, p. 926).

Moreover, from Propositions 1.1 and 1.4 as well as from Theorems 2.1 and 2.4 we get

COROLLARY 2.2. *Let X be an arcwise connected continuum. Then the following conditions are equivalent*:

(i) *X is a dendroid*;

(ii) *every point closed, point connected u.s.c. multifunction F: $X \to X$ has a fixed point*;

(iii) *every refluent multifunction F: $X \to X$ has an almost fixed point.*

For fans and smooth dendroids we obtain (cf. Example 2.2, the multifunction F)

COROLLARY 2.3. *If F: $X \to X$ is a c.c. closed multifunction from a fan X into itself, then F has a fixed point.*

Proof. Let p be a top of X. Suppose, on the contrary, that F has not a fixed point. It follows from Proposition 1.1 and Theorem 2.4 that there is a maximal (in the order $\leqslant_p$) point $x \neq p$ such that x is a limit of an increasing net $\{x_\sigma\}$ with $x \in F(x_\sigma)$ for each σ. Take a component C_σ of $F(x_\sigma)$ containing x. If py is a maximal arc containing px, then $C = \mathrm{Ls}\{C_\sigma\} \subset xy$. Therefore, the continuum C satisfies the equality $C\cap px = \{x\}$. But $F(x)\cap C \neq \emptyset$ because F is c.c. Since $x \notin F(x)$, we infer from Theorem 2.4 that there is a point z in X such that $xz\cap xa \neq \{x\}$, where $a \in F(x)\cap C$ and z has the same property as x. This contradicts the maximality of x.

COROLLARY 2.4. *If F: $X \to X$ is a c.c. closed multifunction from a smooth dendroid X into itself, then F has a fixed point.*

Proof. We proceed as in the proof of Corollary 2.3 taking only as p a point at which X is smooth. It suffices to check that $C = \mathrm{Ls}\{C_\sigma\}$ has only the point x in common with px. In fact, suppose that $y \in C\cap px$.

Then there is a net $\{y_a\}$ such that $y = \lim\{y_a\}$ and $y_a \in C_{\sigma_a}$ with $\sigma_a \geqslant \sigma$. Since X is smooth at p, we have $\mathrm{Lim}\{py_a\} = py$, but $x \in \mathrm{Lim}\{py_a\}$, a contradiction.

The class of λ-dendroids contains trees and dendroids (see [1], Corollary 11, p. 308). For this class of continua we will prove only the following

THEOREM 2.5. *Let X be a λ-dendroid and let a multifunction $F\colon X \to X$ map continua onto continua. Assume that F has the following property:*

(*) If $s \in F(r) \cap rF(r)$ and $\{R_a\}$ is a decreasing family of continua which contain no fixed point of F and if $r \in R_a \cap F(R_a) \subset R_a \subset rs$ and $R = \bigcap R_a \subset I(r, s)$, then $F(R) \cap R \neq \emptyset$.

Then for each p and $q \in F(p)$ there is a point $x \in X$ such that $x \in F(x)$ and $I(p, x) \subset I(p, q)$ $(px \cap pq \neq I(p, x))$.

Proof. Let K be a subcontinuum of X. Put

$$\mathscr{P}(a, K) = \{ab : I(a, b) \subset K \text{ and } b'F(b') \cap ab \subset I(b, a) \text{ for some } b' \in I(b, a)\}.$$

Then

(9) *If $b \notin F(b)$ and $r \in F(b) \cap bF(b)$, then there is an $x \in br \setminus \{b\}$ such that $bx \in \mathscr{P}(b, I(b, r))$ and either $I(x, b) \cap F(I(x, b)) \neq \emptyset$ or $I(b, x) = I(b, r)$.*

In fact, we can assume that br fails to contain a fixed point of F. There is a descending sequence $\{Q_\beta\}$ of subcontinua of br such that $I(b, r) = \bigcap Q_\beta$ and Q_β contains $I(b, r)$ in its interior with respect to br (see [6], p. 650). If there is a β such that $b \notin F(Q_\beta)$, then $F(Q_\beta) \cap I(b, r) = \emptyset$ because $F(Q_\beta)$ is a continuum containing r. Therefore, there is an $a > \beta$ such that $F(Q_a) \cap Q_a = \emptyset$. Taking $x \in Q_a \setminus I(b, r)$, we get $bx \in \mathscr{P}(b, I(b, r))$ and $I(b, x) = I(b, r)$. If for each β we have $b \in F(Q_\beta)$, then $F(I(b, r)) \cap I(b, r) \neq \emptyset$ by (*). Thus $b \in F(I(b, r)) \cap I(b, r)$ because $F(I(b, r))$ is a continuum containing the point r. Hence

$$I(b, r) \in \mathscr{C} = \{R\colon R \text{ is a continuum such that } b \in R \cap F(R) \subset R \subset I(b, r)\}.$$

By Zorn's lemma and condition (*) there is a minimal element R_0 in $\mathscr{C}$. Since $b \in F(R_0)$, there is a point $x \in R_0 \setminus \{b\}$ such that $b \in F(x)$; the image of the continuum bx under F contains b and r, thus $R_0 = bx$ and for each $y \in I(x, b)$ there is a $z \in bx$ such that $y \in F(z)$. From the minimality of R_0 we infer that $z \in I(x, b)$, whence $I(x, b) \cap F(I(x, b)) \neq \emptyset$, i.e. condition (9) holds.

Now let us suppose that there is no $x \in X$ such that $x \in F(x)$ and $ax \in \mathscr{P}(a, K)$. Then the family $\mathscr{P}(a, K)$ is inductive, i.e.

(10) *If $\{ab_a\}$ is a nested family of elements of $\mathscr{P}(a, K)$ and $b = \lim\{b_a\}$, then ab is an element of $\mathscr{P}(a, K)$ with $ab_a \subset ab$ for each a.*

According to Proposition 1 in [9], p. 61, it suffices to show that there is a point b' in $I(b, a)$ such that $b'F(b') \cap ab \subset I(b, a)$. If $I(b, a) \cap F(I(b, a)) \neq \emptyset$, then the required condition is obviously satisfied. Now we assume that $I(b, a) \cap F(I(b, a)) = \emptyset$ and let $r \in F(I(b, a)) \cap bF(I(b, a))$. Suppose, on the contrary, that $rb \cap ab \neq I(b, a)$. Then $rb \cap ab$ is a continuum such that its interior in ab contains $I(b, a)$. Therefore, we can assume that $b_a \in rb \cap ab$ for each a. If a_0 is fixed, then $ab_{a_0} \in \mathscr{P}(a, K)$ and $I(b_{a_0}, a) \cup b_{a_0}b$ is a continuum. Hence $b_a \in F(I(b_{a_0}, a) \cup b_{a_0}b)$ for each a. Thus

$$b \in F(I(b_a, a) \cup b_a b) \cap (I(b_a, a) \cup b_a b) \quad \text{for each } a.$$

From condition (*) we conclude that if $R_a = I(b_a, a) \cup b_a b$, then $F(\bigcap R_a) \cap \bigcap R_a \neq \emptyset$. But $\bigcap R_a = I(b, a)$, a contradiction.

Condition (10) and Zorn's lemma imply

(11) *If $\mathscr{P}(a, K) \neq \emptyset$ and there is no fixed point x of F with $ax \in \mathscr{P}(a, K)$, then there is a maximal element in $\mathscr{P}(a, K)$.*

Condition (9) and Proposition 1 in [9], p. 61, imply

(12) *If ab is maximal in $\mathscr{P}(a, K)$, then $I(b, a) \cap F(I(b, a)) \neq \emptyset$.*

Consider the following family $\mathscr{W}$ of subcontinua of X: $Q \in \mathscr{W}$ provided (i) $Q \cap F(Q) \neq \emptyset$ and (ii) if K' is a proper subcontinuum of Q and $a'b' \in \mathscr{P}(a', K')$, then $a'b' \subset Q$. From (12) we conclude

(13) *If ab is maximal in $\mathscr{P}(a, K)$, then $I(b, a) \in \mathscr{W}$.*

Let $\{W_a\}$ be a decreasing family of elements of $\mathscr{W}$ none of which contains a fixed point. Then

(14) $W = \bigcap W_a \in \mathscr{W}$.

Obviously, we must only show that $F(W) \cap W \neq \emptyset$. Suppose, on the contrary, that $F(W) \cap W = \emptyset$ and let $b \in W$ and $r \in F(b) \cap bF(b)$. It follows from (9) that $\mathscr{P}(b, I(b, r)) \neq \emptyset$. Conditions (11) and (12) imply that there is an x such that $I(b, x) \subset I(b, r)$ and $F(I(x, b)) \cap I(x, b) \neq \emptyset$. If $I(x, b) \cap W \neq \emptyset$, then $bx \subset W$ because X is hereditarily unicoherent and W is a continuum. Thus we can assume that $I(x, b) \cap W_a = \emptyset$ for each a. If for each a there is a β such that $\beta > a$ and $W_\beta \cap bx$ is not contained in $I(b, x)$, then $I(b, x)$ is a proper subcontinuum of W_a for each a. Since $bx \in \mathscr{P}(b, I(b, x))$ and $W_a \in \mathscr{W}$, we infer that $bx \subset W_a$ for each a. Thus $bx \subset W$, but $I(x, b) \subset bx$ and $I(x, b) \cap F(I(x, b)) \neq \emptyset$. Therefore $W \cap F(W) \neq \emptyset$, a contradiction. Hence we can assume that $W_a \cap bx \subset I(b, x)$ for each a. We also obtain a contradiction if for each a the set $W_a \setminus (W_a \cap bx)$ is nonempty. In fact, if $a \in W_a \setminus (W_a \cap bx)$, then taking a continuum ab' in W_a irreducible between a and $W_a \cap bx$ we conclude that $ax = ab' \cup bx$ is in $\mathscr{P}(a, I(a, b'))$, but $I(a, b')$ is a proper subcontinuum of W_a, thus $ax \subset W_a$. As above, this contradicts the assumption that $F(W) \cap W = \emptyset$.

Finally, we can consider only such W_a for which $W_a \subset I(b, x) \subset I(b, r)$. The set $F(W_a)$ is a continuum such that $F(W_a) \cap W_a \neq \emptyset$ and $r \in F(b) \subset F(W_a)$. Consequently, $b \in W_a \cap F(W_a) \subset W_a \subset I(b, r)$ for each a. Condition $(*)$ implies that $F(W) \cap W \neq \emptyset$, i.e. (14) holds.

Now, let $p \in X \setminus F(p)$ and $q \in F(p)$. By condition (9) there is a b such that $pb \in \mathscr{P}(p, I(p, q))$. Suppose, on the contrary, that there is no $x \in X$ such that $x \in F(x)$ and $px \in \mathscr{P}(p, I(p, q))$. From condition (11) it follows that there is a point c such that pc is maximal in $\mathscr{P}(p, I(p, q))$. But then $I(c, p) \in \mathscr{W}$ (cf. (13)) and $I(c, p)$ does not contain a fixed point of F. Therefore, there is a minimal element Q of $\mathscr{W}$ in $I(c, p)$ by (14) and Zorn's lemma. Since $Q \cap F(Q) \neq \emptyset$, by (9) we obtain $\mathscr{P}(a', K') \neq \emptyset$ for some $a' \in Q$ and a proper subcontinuum K' of Q. Conditions (11) and (13) give a contradiction. The proof of Theorem 2.5 is complete.

As a consequence of Proposition 1.3 and Theorems 2.1 and 2.5 we obtain

COROLLARY 2.5. *Let X be a hereditarily decomposable continuum. Then the following conditions are equivalent:*

(i) *X is a λ-dendroid;*

(ii) *every point connected, point closed u.s.c. multifunction $F: X \to X$ has a fixed point;*

(iii) *every point connected c.c. multifunction $F: X \to X$ has an almost fixed point.*

We also have

COROLLARY 2.6. *If a closed continuum-valued multifunction F maps a metric λ-dendroid X into itself, then F has a fixed point.*

Proof. It suffices to show condition $(*)$ of Theorem 2.5. Assume that $s \in F(r) \cap rF(r)$ and let $\{R_n\}$ be a descending sequence of continua satisfying the assumptions of $(*)$. Then $rs \subset F(R_n)$ for $n = 1, 2, \ldots$ Let $\{r_n\}$ be a sequence such that $\lim \{r_n\} = r$ and $r_n \in rs \setminus I(r, s)$. For $n = 1, 2, \ldots$ we can assume that $r_n \notin R_n$ and take $z_n \in R_n$ with $r_n \in F(z_n)$. Similarly, we can assume that $\{z_n\}$ is convergent, and then

$$z = \lim \{z_n\} \in R = \bigcap_{n=1}^{\infty} R_n.$$

If $R \cap F(R) = \emptyset$, then $r \notin F(z)$. Sets $A_n = \{z_m \colon m \geqslant n\} \cup \{z\}$ are closed, and so are $F(A_n)$ because F is closed. Since $r \in F(A_n) \setminus F(z)$, we infer that $r \in F(z_n)$ for some n. But also $r_n \in F(z_n)$. Since $F(z_n)$ is a continuum, we obtain $z_n \in rr_n \subset F(z_n)$. Thus z_n is a fixed point of F in R_n, a contradiction.

Theorem 2.5 and Corollary 2.5 imply known fixed point theorems for λ-dendroids (see [9], [11], [13], [29]). The following question remains open:

PROBLEM 2.1. Is it true that every refluent multifunction F from a Hausdorff λ-dendroid into itself has an almost fixed point? (**P 1236**)

The positive answer to this problem will solve an open question about the fixed point property for continuous point closed multifunctions on λ-dendroids (see [8], Problem, p. 423). For metric spaces λ-dendroids are contained in the class of treelike continua which fail to have a fixed point property even for single-valued continuous mappings (see [2]), but each treelike continuum is contained in a treelike continuum which has the fixed point property for u.s.c. point closed refluent multifunctions (see [10]).

3. Coincidence points. Using standard methods we obtain

THEOREM 3.1. *Let c.c. multifunctions F and G map a connected Hausdorff space X into a (generalized) arc I. Assume that one of the following conditions holds:*

(i) *F is a point connected surjection;*

(ii) *F and G are both surjections.*

Then there is an $x \in X$ such that $F(x) \cap G(x) \neq \emptyset$, i.e. F and G have a coincidence point.

Proof. Let a, b be nonseparating points of I, let $\leqslant$ be the natural linear order in I from a to b, and let $[c, d]$ stand for a closed interval in this order. Put

$$A = \{x \in X\colon \text{ there is } t \in F(x) \text{ such that } G(x) \subset [t, b]\}$$

and

$$B = \{x \in X\colon \text{ there is } t \in G(x) \text{ such that } F(x) \subset [t, b]\}.$$

Then

(15) *The sets A and B are closed.*

Indeed, let $x = \lim\{x_\sigma\}$ and $x_\sigma \in A$ for each σ. Consider a closed interval $[a_\sigma, b_\sigma]$ which is a closure of some component of $F(x_\sigma)$. We can assume that $G(x_\sigma) \subset [b_\sigma, b]$ because $x_\sigma \in A$, and we can assume that $\mathrm{Ls}\{[a_\sigma, b_\sigma]\} = [a_0, b_0]$ because I is compact. Since F and G are c.c., $F(x) \cap [a_0, b_0] \neq \emptyset$ and every component of $G(x)$ meets $[b_0, b]$. If $t_0 \in F(x) \cap [a_0, b_0]$, then t_0 does not belong to $G(x)$. Therefore $G(x) \subset [t_0, b]$, i.e. $x \in A$. This means that A is closed. Similarly, B is closed.

If $c = \inf F(x)$ (in the order $\leqslant$), then either $c \in F(x)$ or there is a d such that $c < d$ and $[c, d] \setminus \{c\} \subset F(x)$ because F is c.c. The multifunction G has the same property, and so

(16) $X = A \cup B$.

Since F is a surjection, there is an $x_0 \in X$ such that $a \in F(x_0)$ and then $x_0 \in A$; in particular, A is nonempty. Similarly, in case (ii), B is nonempty. Then $A \cap B \neq \emptyset$ by (15) and (16) because X is connected.

Taking $x \in A \cap B$ we obtain $F(x) \cap G(x) \neq \emptyset$. Assume now that B is empty and (i) holds. There is an $x_1 \in X$ such that $b \in F(x_1)$ because F is a surjection. The set $F(x_1)$ is connected and $x_1 \in A$ (by (16)); consequently, there is a t_1 with $G(x_1) \subset [t_1, b] \subset F(x_1)$. The proof of Theorem 3.1 is complete.

From Proposition 1.2 and Theorem 3.1 we obtain easily the Theorem from [28], p. 271 (see [18] for some other results in this direction), and we get

COROLLARY 3.1. *Every arclike continuum has the almost fixed point property for* (i) *c.c. point connected multifunctions and* (ii) *c.c. multivalued surjections.*

If Y is not hereditarily unicoherent or contains a triod (i.e., a continuum T such that $T = A \cup B \cup C$, $A \cap B \cap C = A \cap B = A \cap C = B \cap C$, where A, B, and C are proper subcontinua of T), then it is easy to construct a continuum X and two continuous single-valued mappings f and g from X into Y such that $f(x) \neq g(x)$ for $x \in X$ and $f(X) \subset g(X)$. Therefore, Theorem 3.1 and Corollary 2.1 imply

COROLLARY 3.2. *A continuum Y is a* (*generalized*) *arc if and only if for each continuum X and for each two c.c. point connected multifunctions $F, G\colon X \to Y$ with $F(X) \subset G(X) = \overline{G(X)} \subset Y$ there is a coincidence point.*

The following question remains open:

PROBLEM 3.1. Does there exist an arclike continuum which has no almost fixed point for some refluent multifunction? (**P 1237**)

4. End-continua and end-points. A subcontinuum E of a continuum X is called an *end-continuum* of X provided that for each two points $a, b \in X \setminus E$ there is a continuum K such that $\{a, b\} \subset K \subset X \setminus E$ (see [12]). We say that $F\colon X \to Y$ is *biconnected* if F maps continua onto continua and for each continuum $K \subset Y$ the set $F^{-1}(K)$ is a continuum (see [19], p. 462). We have (cf. [12], Corollary 2)

THEOREM 4.1. *If F is a biconnected multifunction from a λ-dendroid onto itself and E is an end-continuum of a subcontinuum Q in X such that $(Q \setminus E) \cap F(Q \setminus E) \neq \emptyset$, then there is a fixed point of F belonging to $X \setminus E$.*

Proof. It follows from the assumptions that there is a $p \in Q$ such that $pF(p) \subset Q \setminus E$. Let qr be a continuum irreducible between E and $pF(p)$ with $q \in E$ and $r \in pF(p)$. If $F\big(pF(p) \cup I(r, q)\big) \cap I(r, q) = \emptyset$, then we find a fixed point $x \in X$ such that $I(r, x) \subset I\big(r, F(p)\big)$ and $rx \cap rF(p) \neq I(r, x)$ by Theorem 2.5 because F satisfies $(*)$ (see [13], Theorem 3). If $F\big(pF(p) \cup I(r, q)\big) \cap I(r, q) \neq \emptyset$, then we consider a continuum irreducible between $I(r, q)$ and $F^{-1}\big(I(r, q)\big)$ in $I(r, q) \cup pF(p)$, and we find a suitable fixed point of F^{-1} (it is also a fixed point of F) because F^{-1} satisfies $(*)$ by the equality $(F^{-1})^{-1}(K) = F(K)$.

Theorem 4.1 generalizes some results of [12], [16], and [19]. For dendroids, end-continua are exactly end-points. Theorems 2.3 and 2.4 imply other results in this direction, namely:

COROLLARY 4.1. *If $F: X \to X$ is a multifunction from a tree X onto itself such that F and F^{-1} are c.c., then for each end-point e such that $F(e) \neq X$ there is a fixed point $x \neq e$.*

COROLLARY 4.2. *If a multifunction $F: X \to X$ from a dendroid X onto itself is such that F and F^{-1} are both u.s.c., point closed and refluent, then for each end-point e with $F(e) \neq X$ there is a fixed point $x \neq e$.*

REFERENCES

[1] D. P. Bellamy, *Composants of Hausdorff indecomposable continua, a mapping approach*, Pacific Journal of Mathematics 47 (1973), p. 303-309.

[2] — *An example of a tree-like continuum without the fixed point property*, p. 35-38 in: Topology Conference, Warsaw 1978, Warszawa 1980.

[3] R. Engelking, *General topology*, Warszawa 1977.

[4] B. En-Nashef, *Fixed points for multi-valued functions on B-spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 21 (1973), p. 523-529.

[5] Z. Frolík, *Concerning topological convergence of sets*, Czechoslovak Mathematical Journal 10 (1960), p. 168-180.

[6] G. R. Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua*, Pacific Journal of Mathematics 36 (1971), p. 647-658.

[7] K. Kuratowski, *Topology*, Vol. II, Warszawa 1968.

[8] T. Maćkowiak, *Continua having the fixed point property for set-valued mappings*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 23 (1975), p. 421-424.

[9] — *Fixed point property for λ-dendroids*, ibidem 26 (1978), p. 61-64.

[10] — *An embedding theorem for tree-like continua*, Colloquia Mathematica Societatis János Bolyai 8, *Topics in topology*, Budapest 1974.

[11] R. Mańka, *Association and fixed points*, Fundamenta Mathematicae 91 (1976), p. 105-121.

[12] — *End-continua and fixed points*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 23 (1975), p. 761-766.

[13] — *Fixed point theorems for λ-dendroids*, Fundamenta Mathematicae (to appear).

[14] E. Michael, *Topologies on spaces of subsets*, Transactions of the American Mathematical Society 71 (1951), p. 152-182.

[15] S. Mrówka, *On the convergence of nets of sets*, Fundamenta Mathematicae 45 (1958), p. 237-246.

[16] T. B. Muenzenberger and R. E. Smithson, *On equivalence of certain coincidence theorems and fixed point theorems*, ibidem 84 (1974), p. 93-100.

[17] — *Refluent multifunctions on semitrees*, Proceedings of the American Mathematical Society 44 (1974), p. 189-195.

[18] R. H. Rosen, *Fixed points for multivalued functions on snake-like continua*, ibidem 10 (1959), p. 167-173.

[19] H. Schirmer, *Biconnected multifunctions of trees which have an end point as fixed and coincidence*, Canadian Journal of Mathematics 23 (1971), p. 461-467.
[20] J. N. Simone, *Concerning hereditarily locally connected continua*, Colloquium Mathematicum 39 (1978), p. 243-251.
[21] R. E. Smithson, *A fixed point theorem for connected multivalued functions*, American Mathematical Monthly 73 (1966), p. 351-355.
[22] — *Fixed point theorems for certain classes of multifunctions*, Proceedings of the American Mathematical Society 31 (1972), p. 595-600.
[23] A. D. Wallace, *A fixed point theorem for trees*, Bulletin of the American Mathematical Society 5 (1954), p. 992-994.
[24] L. E. Ward, Jr., *Partially ordered topological spaces*, Proceedings of the American Mathematical Society 5 (1954), p. 144-161.
[25] — *A note on dendrites and trees*, ibidem 5 (1954), p. 992-994.
[26] — *Mobs, trees and fixed points*, ibidem 8 (1957), p. 798-804.
[27] — *A fixed point theorem for multivalued functions*, Pacific Journal of Mathematics 8 (1958), p. 921-927.
[28] — *A fixed point theorem*, American Mathematical Monthly 65 (1958), p. 271-272.
[29] — *Characterization of the fixed point property for a class of set-valued mappings*, Fundamenta Mathematicae 50 (1961), p. 159-164.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

Reçu par la Rédaction le 12. 11. 1978;
en version modifiée le 26. 5. 1979