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THE HADAMARD PRODUCT AND RELATED DILATIONS

BY

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IN THE MEMORY OF WŁADYSŁAW BACH (1933-1968)

One of the elementary properties of positive definite functions is that the product of two such functions is again positive definite. This is none other than the classical result of Schur [7] which says that the entrywise product [1] (called the *Hadamard product* or, sometimes, the *Schur product*) of two positive definite matrices is also a positive definite matrix.

Positive definiteness, on the other hand, intervenes in dilation theory. Our goal is to extend the Schur theorem to operator entry matrices and to show to what kind of dilation questions this leads.

The paper is organized as follows: in Section 1 we discuss circumstances under which the Hadamard multiplication preserves positive definiteness. In Section 2 we consider dilatability of the product of two operator kernels whilst in Section 3 we deal with operator functions on involution semigroups. Here the point is that we do not require any unit in the semigroup in question. In the last section we handle two more particular cases: the kernel $\exp K$, where K is an operator kernel, and the function $x \to (I-\varphi(x))^{-1}$ restricted to the open unit ball of a C^*-algebra, where the positive linear map φ is defined. It should be noteworthy that neither the open unit nor the function in question is bounded in there.

1. The Hadamard product of operator matrices. Let H be a complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators in H. Let I stand for the identity operator. We say that an $n \times n$ operator matrix $A = (A_{ij})$, $A_{ij} \in B(H)$, is *positive definite* (in short: PD) if for every $f_1, \ldots, f_n$ in H

$$\sum_{i,j=1}^{n} (A_{ij} f_j, f_i) \geqslant 0.$$

[1] This notion, rather neglected, seems to revive in recent years (see also [3], [5], and [8] for further references).

In particular, this implies $A_{ij} = A_{ji}^*$, $i, j = 1, \ldots, n$. Given two $n \times n$ operator matrices $A = (A_{ij})$ and $B = (B_{kl})$, $A_{ij}, B_{kl} \in B(H)$, we denote by $A * B$ their *Hadamard product*, that is, the $n \times n$ matrix $(A_{ij} B_{ij})$. Suppose both A and B are PD. A natural question arises: is the Hadamard product of two PD matrices a PD matrix?

In the scalar case, the old (1911) Schur's theorem answers positively this question. In a general, operator case, the following necessary condition is at hand:

PROPOSITION 1. *If the Hadamard product $A * B$ of two PD matrices A and B is PD, then A_{ij} commutes with B_{ij} for all i and j.*

Notice that if the circumstances of this proposition happen, then $A * B = B * A$.

In fact, this proposition is a simple consequence of self-adjointness of $A * B$ as well as A and B. This is why the commutativity condition $A_{ij} B_{ij} = B_{ij} A_{ij}$ does not become sufficient as we show by an example.

Example 1. We wish to present a PD operator matrix whose Hadamard square is not PD. Take two (strictly) positive operators M and N, necessarily non-commuting, such that $\|M^{-1} NMN^{-1}\| > 1$. Define

$$A = \begin{pmatrix} M^2 & MN \\ NM & N^2 \end{pmatrix}.$$

Then

$$A * A = \begin{pmatrix} M^4 & (MN)^2 \\ (NM)^2 & N^4 \end{pmatrix}.$$

It is easy to check that A is PD but $A * A$ is not. To see this, it could be convenient to use the following criterion [1]: a matrix

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

with A and B positive is PD if and only if there is a contraction W such that $C = A^{1/2} W B^{1/2}$.

A look at Schur's proof (or rather its Halmos' version [4]) suggests the following sufficient condition:

PROPOSITION 2. *Given two $n \times n$ operator matrices $A = (A_{ij})$ and $B = (B_{ij})$ with B being PD and such that A_{ij} commutes with B_{ij} for all i and j. Suppose, moreover, that each entry A_{ij} is of the form*

$$A_{ij} = \sum_{\alpha} C_{\alpha i}^* C_{\alpha j}, \tag{1}$$

$C_{\alpha j} \in B(H)$ and $C_{\alpha j}$ commutes with B_{ij} for $i \leqslant j$ and α arbitrary. Then $A * B$ is PD.

Proof. Take $f_1, \ldots, f_n$ in H and write

$$\sum_{i,j}(A_{ij}B_{ij}f_j, f_i) = \sum_{i\leqslant j}(A_{ij}B_{ij}f_j, f_i) + \sum_{i>j}(B_{ij}A_{ij}f_j, f_i)$$
$$= \sum_{\alpha}\Big(\sum_{i\leqslant j}(C^*_{\alpha i}C_{\alpha j}B_{ij}f_j, f_i) + \sum_{i>j}(B_{ij}C^*_{\alpha i}C_{\alpha j}f_j, f_i)\Big).$$

Since $C_{\alpha j}$ commutes with B_{ij} for $i \leqslant j$, $C^*_{\alpha j}$ commutes with $B_{ji} = B^*_{ij}$ for $i < j$ or, equivalently, $C^*_{\alpha i}$ commutes with B_{ij} for $i > j$. Thus the first ingredient of the right-hand side is equal to

$$\sum_{\alpha}\sum_{i\leqslant j}(B_{ij}C_{\alpha j}f_j, C_{\alpha i}f_i)$$

and the second one to

$$\sum_{\alpha}\sum_{i>j}(B_{ij}C_{\alpha j}f_j, C_{\alpha i}f_i).$$

Putting both these sums together we have

$$\sum_{i,j}(A_{ij}B_{ij}f_j, f_i) = \sum_{\alpha}\sum_{i,j}(B_{ij}C_{\alpha j}f_j, C_{\alpha i}f_i) \geqslant 0.$$

Example 2(2). Suppose $A = (A_{ij})$ is of the form

$$A = \begin{pmatrix} I & U & & U^{n-1} \\ U^* & I & & U^{n-2} \\ \dots & \dots & \dots & \dots \\ U^{*n-1} & U^{*n-2} & \dots & I \end{pmatrix}$$

with U being an isometry and $B = (B_{ij})$ is an arbitrary PD matrix. Suppose, moreover, U commutes with B_{ij} for $i \leqslant j$. Since $A_{ij} = U^{*i}U^j$ and, consequently, A is of the form (1), where $C_{\alpha j} = U^j$ commutes with B_{ij} for $i \leqslant j$, we can just use Proposition 2 to get in conclusion that

$$A*B = \begin{pmatrix} B_{11} & \dots & U^{n-1}B_{1n} \\ \dots & \dots & \dots \\ U^{*n-1}B_{n1} & \dots & B_{nn} \end{pmatrix}$$

is a PD matrix.

A less subtle version of Proposition 2 is the following

PROPOSITION 3. *Suppose A is of the form* (1) *and $C_{\alpha j}$ commutes with B_{ij} for all i, j, and α. If B is PD, then so is $A*B$.*

The proof goes as follows:

$$\sum_{i,j}(A_{ij}B_{ij}f_j, f_i) = \sum_{\alpha}\sum_{i,j}(C^*_{\alpha i}C_{\alpha j}B_{ij}f_j, f_i) = \sum_{\alpha}\sum_{i,j}(B_{ij}C_{\alpha j}f_j, C_{\alpha i}f_i) \geqslant 0.$$

(2) The author would like to thank D. Timotin for a remark improving the first version of this example.

Another version of the above is (cf. [11])

PROPOSITION 4. *Suppose A and B are such that each entry A_{ij} commutes with each entry B_{kl}. Then $A*B$ is PD.*

This follows from Proposition 3 and elementary spectral theory. Since A acts as a positive operator on the direct sum of n copies of H, we have $A^{1/2} = (C_{\alpha j})$. Then $C_{\alpha i}^* = C_{i\alpha}$ and A_{ij} takes the form (1). The $(n \times n)$-matrix $\operatorname{diag}(B_{kl}, \ldots, B_{kl})$ commutes with A, so does it with $A^{1/2}$. This implies that $C_{\alpha i}$ commutes with B_{kl} for all α, i, k, and l. Now, an application of Proposition 3 completes the proof.

In the sequel we will make use just of Proposition 4.

2. Dilation of operator kernels. The most general setup of dilation theory is as follows: suppose X is a set and $K\colon X \times X \to B(H)$, called an *operator kernel* on X, is PD, that is, for every finite number of x's say, $x_1, \ldots, x_n$, the matrix $(K(x_i, x_j))$ is PD. It is known [6] that K factors as

$$K(x, y) = F(x)^* F(y), \qquad x, y \in X, \tag{2}$$

where $F\colon X \to B(H_K)$ with an appropriate Hilbert space H_K.

Suppose we are given a semigroup S of actions on X. Denote the action of $s \in S$ on $x \in X$ by sx. It is reasonable to ask whether there exists a multiplication-preserving map $\Phi\colon S \to B(H_K)$ such that $F(sx) = \Phi(s)F(x)$, $s \in S$, $x \in X$. Then the factorization (2) takes the form

$$K(sx, ty) = F(x)^* \Phi(s)^* \Phi(t) F(y), \tag{3}$$

$s, t \in S$, $x, y \in X$. One can easily prove [6] that such a Φ exists if and only if K satisfies the boundedness condition

$$\sum_{i,j} (K(sx_i, sx_j) f_j, f_i) \leqslant c(s) \sum_{i,j} (K(x_i, x_j) f_j, f_i) \tag{BC}$$

for $x_1, \ldots, x_n$ in X, $s \in S$, and $f_1, \ldots, f_n$ in H. In case X is a semigroup (with unit) itself, $S = X$ with sx being just the semigroup product of s and x, the dilation character of (3) becomes more transparent. We have

$$K(s, t) = R^* \Phi(s)^* \Phi(t) R, \qquad s, t \in S,$$

where $R = F(1)$ is a fixed operator (being an isometry if $K(1, 1) = I$).

Now, suppose we have two operator kernels K_i on X_i ($i = 1, 2$). Define the kernel L on $X_1 \times X_2$ by

$$L(x_1, x_2, y_1, y_2) = K_1(x_1, y_1) K_2(x_2, y_2).$$

Suppose

(*) $K_1(x_1, y_1)$ commutes with $K_2(x_2, y_2)$ for all x_1, y_1, x_2, y_2.

Then, as an immediate consequence of Proposition 4, we get

(**) L is a PD kernel on $X_1 \times X_2$.

Suppose we are given two semigroups S_1 and S_2 acting on X_1 and X_2, respectively. Then $S_1 \times S_2$ acts in a natural way on $X_1 \times X_2$ by $(s_1, s_2)(x_1, x_2) = (s_1 x_1, s_2 x_2)$. Suppose, furthermore, both K_i satisfy (BC) with $c_i(s)$. A question arises: does the kernel L satisfy (BC)? To answer this, note that (BC) can be read as follows: for every $s \in S$ there is $c(s) \geqslant 0$ such that the kernel $c(s)K - K^s$, where $K^s(x, y) = K(sx, sy)$, is PD. Write

$$\begin{aligned} & c_1(s_1)c_2(s_2)K_1(x_1, y_1)K_2(x_2, y_2) - K_1^{s1}(x_1, y_1)K_2^{s2}(x_2, y_2) \\ & \quad = \big(c_1(s_1)K_1(x_1, y_1) - K_1^{s1}(x_1, y_1)\big)\big(c_2(s_2)K_2(x_2, y_2) - K_2^{s2}(x_2, y_2)\big) + \\ & \qquad + c_1(s_1)K_1(x_1, y_1)\big(c_2(s_2)K_2(x_2, y_2) - K_2^{s2}(x_2, y_2)\big) + \\ & \qquad + c_2(s_2)\big(c_1(s_1)K_1(x_1, y_1) - K_1^{s1}(x_1, y_1)\big)K_2(x_2, y_2). \end{aligned}$$

Since the terms on the right-hand side are PD, so is the left-hand side (we use here again Proposition 4). Invoking the remark on (BC) we have just made, we get the answer to our question:

(***) L satisfies (BC) provided so do both K_i.

Notice that L satisfies (BC) with $c_1(s_1)c_2(s_2)$. Putting (**) and (***) together we have

$$K_1(s_1 x_1, t_1 y_1)K_2(s_2 x_2, t_2 y_2) = F(x_1, x_2)^* \Phi(s_1, s_2)^* \Phi(t_1, t_2) F(y_1, y_2),$$

where $F\colon X_1 \times X_2 \to B(H_{K_1K_2})$ and $\Phi\colon S_1 \times S_2 \to B(H_{K_1K_2})$ is a homomorphism. Since $(s_1, s_2) = (s_1, 1)(1, s_2) = (1, s_2)(s_1, 1)$, Φ factors as $\Phi(s_1, s_2) = \Phi_1(s_1)\Phi_2(s_2)$ with $\Phi_1(s_1)$ commuting with $\Phi_2(s_2)$. Finally, we get the following

THEOREM 1. *Suppose K_i ($i = 1, 2$) are PD kernels on semigroups S_i, each S_i has a unit. Suppose, moreover, K_i satisfy* (*) *and* (BC). *Then there are another Hilbert space $H_{K_1K_2}$ and semigroup homomorphisms $\Phi_1\colon S_1 \to B(H_{K_1K_2})$ and $\Phi_2\colon S_2 \to B(H_{K_1K_2})$ such that*

$$K_1(s_1, t_1)K_2(s_2, t_2) = R^* \Phi_2(s_2)^* \Phi_1(s_1)^* \Phi_1(t_1)\Phi_2(t_2) R,$$

where $s_1, t_1 \in S_1$, $s_2, t_2 \in S_2$, and $R\colon H \to H_{K_1K_2}$ is a bounded linear operator. Moreover, $\Phi_1(s_1)$ commutes with $\Phi_2(s_2)$ for every s_1 and s_2.

3. Dilation on involution semigroups. In case S is an involution semigroup consider a function $\varphi\colon S \to B(H)$. Call it a *PD function* if the kernel $K(s, t) = \varphi(s^* t)$ is PD. In this case the dilation of φ is an involution-preserving homomorphism Φ of S into $B(H_\varphi)$ such that

$$\varphi(s) = R^* \Phi(s) R, \quad s \in S, \tag{4}$$

with a bounded linear operator R.

We can apply all what we have shown so far and get in this way the result of [11]. But we can proceed one step further. Drop the usual requirement that S has a unit and look what will happen. The suitable dilation theorem, proved in [10], states that $\varphi\colon S\to B(H)$ is dilatable if and only if φ satisfies (instead of positive definiteness) the condition

$$\text{(EX)}\qquad \sum_{i,j}(\varphi(s_i^* s_j)f_j, f_i)\geqslant C\Big\|\sum_i \varphi(s_i)f_i\Big\|^2 \quad\text{and}\quad \varphi(s^*)=\varphi(s)^*$$

with $s, s_1, \ldots, s_n$ in S, $f_1, \ldots, f_n$ in H and $C>0$, and, in addition, φ satisfies (BC).

As before, we deal with two semigroups S_1 and S_2, now being $*$-semigroups and two operator functions φ_1 and φ_2 on them. Suppose each $\varphi_1(s_1)$ commutes with each $\varphi_2(s_2)$.

We know that, by (***), the function $\varphi\colon S_1\times S_2\to B(H)$ defined as $\varphi(s_1, s_2)=\varphi_1(s_1)\varphi_2(s_2)$, $s_1\in S_1$, $s_2\in S_2$, satisfies (BC) provided φ_1 and φ_2 do so. But what about (EX)? The idea is again to interpret the first part of (EX) as positive definiteness of some kernel.

Define

$$K_i(s_i, t_i)=\varphi_i(s_i^* t_i)-C_i\varphi_i(s_i)^*\varphi_i(t_i)$$

and

$$L(s_1, s_2, t_1, t_2)=\varphi_1(s_1^* t_1)\varphi_2(s_2^* t_2)-C_1C_2\varphi_2(s_2)^*\varphi_1(s_1)^*\varphi_1(t_1)\varphi_2(t_2).$$

All what we have to show is that L is PD on $S_1\times S_2$ provided so are both K_i. With the above notation we have (making use of commutativity of φ_1 and φ_2)

$$L(s_1, s_2, t_1, t_2)=\varphi_1(s_1^* t_1)K_2(s_2, t_2)+\varphi_2(s_2)^*\varphi_2(t)K_1(s_1, t_1).$$

Since $\varphi_1(s_1^* t_1)$ commutes with $K_2(s_2, t_2)$ and so does $\varphi_2(s_2)^*\varphi_2(t_2)$ with $K_1(s_1, t_1)$ and since all four involved kernels $(s_1, t_1)\to\varphi_1(s_1^* t_1)$, K_2, $(s_2, t_2)\to\varphi_2(s_2)^*\varphi_2(t_2)$, and K_1 are PD, appealing to (**) we infer that the right-hand side is PD as well. Thus we get the first part of (EX). The second is obvious.

Summarizing, we have proved the following

THEOREM 2. *Suppose we are given two functions $\varphi_i\colon S_i\to B(H)$ $(i=1, 2)$ satisfying* (EX) *and* (BC). *Suppose each operator $\varphi_1(s_1)$ commutes with each $\varphi_2(s_2)$. Then*

$$\varphi_1(s_1)\varphi_2(s_2)=R^*\Phi_1(s_1)^*\Phi_2(s_2)R,\qquad s_1\in S_1,\ s_2\in S_2,$$

where $\Phi_i\colon S_i\to B(K_{\varphi_1\varphi_2})$ is a $*$-homomorphism and $R\colon H\to H_{\varphi_1\varphi_2}$ is a bounded linear operator.

Recall we do not require any unit in the $*$-semigroups S_i.

4. Two special cases. (a) Suppose S is a semigroup with unit and K is an operator kernel on it. Take $s, t\in S$ and define the operator

$$\exp K(s, t)=\sum_{n=0}\frac{1}{n!}K(s, t)^n$$

and the kernel

$$\exp K\colon (s, t) \to \exp K(s, t).$$

Suppose K has a dilation. The question is: does $\exp K$ share this property with K? Assuming the range of K, that is, the set $\{K(s, t)\colon s, t \in S\}$, to be commutative, the answer is: yes. This is due to Theorem 1. To see this we repeatedly apply Theorem 1 to each $K(s, t)^n$. More precisely, since $K(s, t)^n = K(s, t)^{n-1} K(s, t)$, we apply Theorem 1 with $K_1 = K^{n-1}$ and $K_2 = K$ and restrict it to the diagonal of $S \times S$, that is, to $\{(s, s)\colon s \in S\}$ which is a subsemigroup of $S \times S$.

All these enable us to formulate the following

COROLLARY 1. *Suppose the operator kernel K on a semigroup with unit is PD and satisfies* (BC). *Suppose, moreover, the range of K forms a commutative set of operators. Then there is a semigroup homomorphism $\Phi\colon S \to B(H_K)$ such that*

$$\exp K(s, t) = R^* \Phi(s)^* \Phi(t) R, \qquad s, t \in S,$$

where $R\colon H \to H_K$ is a bounded linear operator.

(b) This example is more involved. Suppose A is a C^*-algebra with unit. It belongs to rudiments of theory of C^*-algebras that if f is a positive linear functional on A, then

$$|f(y^* xy)| \leqslant \|x\| f(y^* y), \qquad x, y \in A. \tag{5}$$

Let $\varphi\colon A \to B(H)$ be a positive linear map such that $\varphi(1) = I$. Denote by S the (open) unit ball in A, that is, $S = \{x\colon \|x\| < 1\}$. Now, S becomes a (multiplicative) involution semigroup. Fix $x \in S$; then, by (5), $\|\varphi(x)\| < 1$ and the operator $(I - \varphi(x))^{-1}$ exists and

$$(I - \varphi(x))^{-1} = I + \varphi(x) + (\varphi(x))^2 + \ldots \tag{6}$$

Set $\psi(x) = (I - \varphi(x))^{-1}$ for $x \in S$. We are interested in dilatability of $\psi\colon S \to B(H)$. The absence of unit in S makes it impossible to use the well-known Sz.-Nagy dilation theorem ([12], Principal Theorem). But we can use our, non-unital, version quoted (from [10]) in Section 3. We prove the following

COROLLARY 2. *Suppose φ is a positive linear map of a C^*-algebra A, $\varphi(1) = I$. Define S as above. If the range of φ is commutative, then there exist another Hilbert space H_φ and an involution-preserving homomorphism $\Phi\colon S \to B(H_\varphi)$ such that*

$$(I - \varphi(s))^{-1} = V^* \Phi(s) V, \qquad s \in S,$$

where V is an isometry of H into H_φ.

Proof. Since the range of φ is commutative, it is in fact in a commutative C^*-algebra (here we use the equality $\varphi(s^*) = \varphi(s)^*$) and, by [2], Proposition 1.2.2, φ is completely positive, which in our terminology, means φ is a PD map. Then, by [9], φ must satisfy (EX) with $C = 1$. Inequality (5) yields that φ satisfies (BC) with $c(x) = \|x\|^2$. Now we apply Theorem 2 to each ingredient on the right-hand side of (6) as we have done this in case (a) of this section. In this way we infer that ψ satisfies condition (EX) with $C = 1$ and also condition (BC) with

$$c(x) = 1 + \|x\|^2 + \|x\|^4 + \ldots = (1 - \|x\|^2)^{-1}.$$

Thus ψ has a dilation Φ and $\|\Phi(x)\| \leqslant (1 - \|x\|^2)^{-1/2}$. Moreover, V is an isometry. This can be easily obtained from what has been established in [9] and we omit more detailed argumentation.

Note. This is a substantially enlarged version of our earlier paper *The Hadamard product of dilatable operator kernels* which has been circulating, as a preprint, since June 1977.

Result (b) of Section 4 was presented, as a short communication, to the International Congress of Mathematicians, Helsinki, August 1978.

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