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THE CONSTRUCTION OF RESOLVABLE BLOCK-SYSTEMS

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Let

$$R = \left\{ B_i \colon i = 1, 2, \ldots, \lambda \frac{v(v-1)}{k(k-1)} \right\}$$

be a family of k-element subsets (called *blocks*) of a v-element set V. We say that R is an $\mathrm{RB}(v, k, \lambda)$ if it has the following two properties:

1° Each pair (i.e., two-element subset) of elements of V is contained in exactly λ blocks of R.

2° The family R can be partitioned into $\lambda(v-1)/(k-1)$ subfamilies R_j ($j = 1, 2, \ldots, \lambda(v-1)/(k-1)$) in such a way that each R_j consists of v/k disjoint blocks B_s^j ($s = 1, 2, \ldots, v/k$).

The subfamilies R_j of R are called *parallel classes* and R is called a *resolvable block systems.*

The existence of $\mathrm{RB}(v, 3, 1)$ for $v \equiv 3 \pmod 6$, $\mathrm{RB}(v, 3, 2)$ for $v \equiv 0 \pmod 3$ ($v \neq 6$), $\mathrm{RB}(v, 4, 1)$ for $v \equiv 4 \pmod{12}$, and $\mathrm{RB}(v, 4, 3)$ for $v \equiv 4$ or $8 \pmod{12}$ was shown in [1] and [3]-[5].

In this paper we present a construction of $\mathrm{RB}(2p^2, 2p, 2p-1)$, where p is a power of a prime number.

We recall some well-known definitions:

1. Let P be the set of all pairs of elements of $I = \{1, 2, \ldots, 2r\}$. A family $\{P_1, P_2, \ldots, P_{2r-1}\}$ of subsets of P is called a *resolvable pair system* $\mathrm{RP}(2r)$ if $|P_i| = r$ ($i = 1, 2, \ldots, 2r-1$), $P_i \cap P_j = \emptyset$ for $i \neq j$, and $P_1 \cup P_2 \cup \ldots \cup P_{2r-1} = P$. It is known that $\mathrm{RP}(2r)$ exists for every r. Each P_i in $\mathrm{RP}(2r)$ is called a *parallel class* of the system of pairs.

2. Let $G_1, G_2, \ldots, G_m$ and $H_1, H_2, \ldots, H_n$ be partitions of a set E, $|E| = mn$, such that $|G_i| = n$, $|H_j| = m$, and $|G_i \cap H_j| = 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Consider a family $F = \{F_1, \ldots, F_{n^2}\}$ of m-element subsets of E satisfying the following conditions:

(a) $|F_s \cap G_i| = 1$ for $i = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n^2$;

(b) each pair $\{x, y\}$ with $x \in G_i$, $y \in G_j$, $i \neq j$, is contained in exactly one set in F;

(c) the family F can be partitioned into n subfamilies (parallel classes), each consisting of n disjoint blocks.

Such a family F is called a *resolvable transversal design* and will be denoted by $\mathrm{RTD}(m, n)$. If $m \leqslant n$ and n is a power of a prime number, then $\mathrm{RTD}(m, n)$ exists (see [2]).

Now we shall construct $\mathrm{RB}(2p^2, 2p, 2p-1)$, where p is a power of a prime number. Let

$$V = \{1, 2, \ldots, 2p^2\} \quad \text{and} \quad R_1 = \{B_0, B_1, \ldots, B_{p-1}\},$$

where $B_s = \{2sp+1, 2sp+2, \ldots, 2sp+2p\}$ for $s = 0, 1, \ldots, p-1$. On each block B_s we construct a resolvable pair system

$$\mathrm{RP}_s(2p) = \{P_{s,1}, P_{s,2}, \ldots, P_{s,2p-1}\}.$$

A triple (s, j, i) stands for the j-th pair in $P_{s,i}$. The set of all triples (s, j, i) with fixed i will be denoted by E_i. Clearly, $|E_i| = p^2$. We define two partitions of E_i:

$$G_s = \{(s, j, i)\colon j = 1, 2, \ldots, p\} \quad (s = 0, 1, \ldots, p-1)$$

and

$$H_j = \{(s, j, i)\colon s = 0, 1, \ldots, p-1\} \quad (j = 1, 2, \ldots, p).$$

Obviously, $|G_s \cap H_j| = 1$. Since p is a power of a prime, there exists an $\mathrm{RTD}(p, p)$, say F^i, on E_i. Now F^i can be partitioned into p parallel classes. If in each set in a parallel class the triples (s, j, i) are replaced by pairs of elements V represented by these triples, then the unions of such pairs give $2p$-element subsets of V. Indeed, different triples in a parallel class determine disjoint pairs.

In this manner, partitions of each F^i into p parallel classes determine $p(p-1)$ families of $2p$-element subsets of V. Those families will be denoted by $R_p, \ldots, R_{2p^2-1}$. Let $R_{p-1} = R_{p-2} = \ldots = R_2 = R_1$, so that we obtain $2p^2-1$ families of $2p$-element subsets of V and each R_i consists of p such subsets.

We show that $R = \{R_1, R_2, \ldots, R_{2p^2-1}\}$ is an $\mathrm{RB}(v, k, \lambda)$, where $v = 2p^2$, $k = 2p$, and $\lambda = 2p-1$.

Clearly, the blocks in R are of cardinality $k = 2p$ and there are exactly $\lambda v(v-1)/k(k-1) = p(2p^2-1)$ blocks. Moreover, from the definition of R_i it follows immediately that condition 2° is satisfied.

Now we show that 1° holds with $\lambda = 2p-1$. We consider 2 cases:

(i) If $\{x, y\}$ is contained in $B_s \in R_1$, then it occurs in exactly $p-1$ blocks in $R_1, \ldots, R_{p-1}$. Moreover, to this pair there corresponds a triple (s, j, i) which belongs to exactly one E_i. From the properties of $\mathrm{RTD}(p, p)$ built on E_i it follows that the triple, and so the pair $\{x, y\}$, occurs in exactly p blocks $\mathrm{RTD}(p, p)$. In this case $\{x, y\}$ occurs in exactly $2p-1$ blocks $\mathrm{RB}(2p^2, 2p, 2p-1)$.

(ii) Let the elements of $\{x, y\}$ belong to different blocks of R_1, e.g., $x \in B_s$ and $y \in B_r$, $s \neq r$. For each $i = 1, 2, \ldots, 2p-1$ we consider the set E_i and the corresponding RTD(p, p). The cells G_s and G_r contain the triples (s, j_1, i) and (r, j_2, i), respectively. Those triples correspond to pairs $\{x, t\}$ and $\{z, y\}$ and both triples (s, j_1, i) and (r, j_2, i) are contained in some block of the parallel class RTD(p, p) on E_i. Therefore, the set $\{x, y, t, z\}$, and so the pair $\{x, y\}$, is contained in exactly one block occurring in the families $R_p, \ldots, R_{2p^2-1}$. Since i takes on the values $1, 2, \ldots, 2p-1$, the pair $\{x, y\}$ occurs in exactly $2p-1$ blocks RB$(2p^2, 2p, 2p-1)$, which completes the proof of correctness of the construction.

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