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EXISTENCE OF GENERALIZED AFFINE SYMMETRIC SPACES OF ARBITRARY ORDER

BY

MILOŠ BOŽEK (BRATISLAVA)

In [1] the following existence theorem on generalized symmetric Riemannian spaces was proved:

THEOREM A. *For every even integer $n \geqslant 4$ there is a generalized symmetric Riemannian space of order n diffeomorphic to R^{n-1} and such that the identity component of its full isometry group is solvable.*

The main result of the present paper is an affine counterpart of this theorem:

THEOREM 1. *For every integer $n \geqslant 3$ there is a generalized affine symmetric space of order n diffeomorphic to R^{2n-2} and such that the identity component of its full affine group is solvable.*

1. Generalized affine symmetric spaces. All differentiable manifolds, mappings, and tensor fields are supposed to be of class C^∞.

Let (M, ∇) be a differentiable affine manifold and let $x \in M$ be a point. A *symmetry* of (M, ∇) at x is an affine transformation s_x of (M, ∇) for which x is an isolated fixed point. An *s-structure* on (M, ∇) is a family $\{s_x\}_{x\in M}$ of symmetries of (M, ∇), briefly denoted by $\{s_x\}$. To every s-structure $\{s_x\}$ on (M, ∇) there corresponds a tensor field S on M of type $(1, 1)$ defined by $S_x = (s_x)_{*,x}$ for all $x \in M$. The tensor field S is called the *symmetry tensor field* of $\{s_x\}$.

Following Kowalski [5], a *generalized affine symmetric space* (briefly, a g.a.s. space) is a connected affine manifold (M, ∇) admitting an s-structure $\{s_x\}$ with the following properties:

(i) The map $M\times M \to M$, $(x, y) \mapsto s_x(y)$, is differentiable.

(ii) For every $x, y \in M$, $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$.

(iii) The symmetry tensor field S of $\{s_x\}$ is parallel, i.e., $\nabla S = 0$.

The s-structure $\{s_x\}$ on (M, ∇) satisfying conditions (i)-(iii) is called an *admissible s-structure*.

Property (iii) of admissible s-structures yields the following

PROPOSITION 1.1. *For every symmetry s_o' of (M, ∇) at a point $o \in M$ there is at most one admissible s-structure $\{s_x\}$ on (M, ∇) such that $s_o = s_o'$.*

An s-structure $\{s_x\}$ on (M, ∇) is said to be of *order* k if k is the least positive integer for which $(s_x)^k = \mathrm{id}$ for all $x \in M$. If there is a point $x \in M$ such that $(s_x)^k \neq \mathrm{id}$ for all integers k, then $\{s_x\}$ is said to be of *order* ∞. The *order of a g.a.s. space* (M, ∇) is the least element k of the set $\{2, 3, \ldots \ldots, \infty\}$ for which there is an admissible s-structure of order k on (M, ∇).

In contrast to the Riemannian case, there are g.a.s. spaces of order ∞ (cf. [3], Theorem 2, and [6]). Generalized affine symmetric spaces of order 2 are nothing but the affine symmetric spaces. (Recall that, in this case, there is a unique admissible s-structure $\{s_x\}$ of order 2 — it consists of usual geodesic symmetries and $S = -\mathrm{id}$.) Thus, g.a.s. spaces of order $k \geqslant 3$ are those of special interest.

An *automorphism* of an s-structure $\{s_x\}$ on (M, ∇) is a diffeomorphism $f\colon M \to M$ such that

$$f \circ s_x = s_y \circ f, \quad y = f(x), \quad \text{for each } x \in M.$$

If $\{s_x\}$ is an admissible s-structure, then property (ii) implies that every symmetry s_x is an automorphism of the s-structure $\{s_x\}$. The following is the easier part of a basic theorem proved in [4] (cf. [5], Theorem B):

THEOREM B. *Let $\{s_x\}$ be an admissible s-structure on a g.a.s. space (M, ∇). Then:*

1. *The group* $\mathrm{Aut}(\{s_x\})$ *of all automorphisms of the s-structure $\{s_x\}$ is a transitive Lie transformation group of M, which is a closed subgroup of the full affine transformation group $A(M, \nabla)$.*

2. *If G is the identity component of* $\mathrm{Aut}(\{s_x\})$, *$o \in M$ is a fixed point, and G_o the isotropy subgroup of G at o, then the homogeneous space $M = G/G_o$ is reductive in a canonical way and the connection ∇ is the canonical connection on G/G_o.*

COROLLARY. (a) *Every g.a.s. space is a complete affine manifold.*

(b) *The connection ∇ of every g.a.s. space (M, ∇) is parallel, i.e., ∇ has parallel curvature and torsion: $\nabla R = 0$ and $\nabla T = 0$.*

2. Admissible automorphisms of transvection algebra. In this section we construct an infinitesimal analogy to every admissible s-structure on a simply connected and connected affine manifold (M, ∇). Necessary conditions for the existence of admissible s-structures are given in the Corollary to Theorem B. Hence throughout this section we assume that (M, ∇) always denotes a simply connected and connected affine manifold with complete parallel connection.

We shall use the following notation: o is a fixed point of M; $\mathfrak{m}$ is the tangent vector space $T_o(M)$; K is the isotropy subgroup of the full affine group $A(M, \nabla)$ at o; $A(M, \nabla)^0$ and K^0 are the identity components; $\mathfrak{a}$ and $\mathfrak{k}$ denote the Lie algebras of the groups $A(M, \nabla)$ and K, respectively; $\hat{\mathfrak{k}}$ is the subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{m})$ given by

$$\hat{\mathfrak{k}} = \{A \in \mathfrak{gl}(\mathfrak{m}) | A(R_o) = A(T_o) = 0\}.$$

First we give a description of the algebra $\mathfrak{a}$. For this purpose let us consider the Lie algebra $\hat{\mathfrak{a}} = \hat{\mathfrak{k}} + \mathfrak{m}$ (vector space direct sum) with the bracket operation defined as follows:

$$(1) \qquad \begin{aligned} [X, Y] &= \big(-R_o(X, Y), -T_o(X, Y)\big), \\ [A, X] = A(X), &\qquad [A, B] = A \circ B - B \circ A \end{aligned}$$

for all $A, B \in \hat{\mathfrak{k}}$ and $X, Y \in \mathfrak{m}$. The assumptions $\nabla R = \nabla T = 0$ imply that $R_o(X, Y)(R_o) = R_o(X, Y)(T_o) = 0$ for every $X, Y \in \mathfrak{m}$. Consequently, every curvature transformation $R_o(X, Y)$, X, $Y \in \mathfrak{m}$, is an element of $\hat{\mathfrak{k}}$. This means that the bracket operation in $\hat{\mathfrak{a}}$ is well defined. Using the basic properties of curvature and torsion we can easily see that $\hat{\mathfrak{a}}$ is actually a Lie algebra.

As is well known, the assumptions on (M, ∇) imply that the group $A(M, \nabla)^0$ is transitive on M and that K^0 is the isotropy subgroup of $A(M, \nabla)^0$ at o. By [2], Theorem X.2.8, the homogeneous space $A(M, \nabla)^0/K^0$ is reductive in a natural way and, under the standard identification $M = A(M, \nabla)^0/K^0$, ∇ is the canonical connection on $A(M, \nabla)^0/K^0$. Therefore, we can write $\mathfrak{a} = \mathfrak{k} + \mathfrak{m}$ (vector space direct sum). By Corollary VI.7.9 of [2], the linear isotropy representation λ of K in $\mathfrak{m}$ maps K isomorphically onto the group $\hat{K}$ consisting of all automorphisms ϕ of the vector space $\mathfrak{m}$ such that $\phi(R_o) = R_o$ and $\phi(T_o) = T_o$. Clearly, the Lie algebra of the group $\hat{K}$ is the algebra $\hat{\mathfrak{k}}$. Hence the induced representation $\lambda_*: \mathfrak{k} \to \hat{\mathfrak{k}}$ is a Lie algebra isomorphism. As is well known, λ_* is the restriction of the adjoint representation of $\mathfrak{k}$ in $\mathfrak{a}$ to $\mathfrak{m}$. Using this fact and formulas (1) and (2) of [2], Theorem X.2.6, we obtain the following

PROPOSITION 2.1. *The map $\varphi = \lambda_* + \mathrm{id}_{\mathfrak{m}}$ is an isomorphism between the Lie algebras $\mathfrak{a}$ and $\hat{\mathfrak{a}}$.*

In the sequel, the algebras $\mathfrak{a}$ and $\hat{\mathfrak{a}}$ will be identified by the isomorphism φ.

Now, let $\hat{\mathfrak{h}}$ be the vector subspace of the Lie algebra $\mathfrak{gl}(\mathfrak{m})$ generated by the set of all curvature transformations $R_o(X, Y)$, X, $Y \in \mathfrak{m}$. It follows immediately from (1) that $\hat{\mathfrak{h}} \subset \hat{\mathfrak{k}}$. By [5], Theorem 2, the vector space

$\mathfrak{t} = \hat{\mathfrak{h}} + \mathfrak{m}$ is a subalgebra of the Lie algebra $\hat{\mathfrak{a}}$. With respect to the geometrical interpretation of the algebra $\mathfrak{t}$ given in the same theorem, $\mathfrak{t}$ is called the *transvection algebra* of (M, ∇).

Every admissible s-structure $\{s_x\}$ on (M, ∇) induces a linear transformation ϕ of the transvection algebra $\mathfrak{t}$ defined by $\phi = \mathrm{id} + S_o$, where id is the identity transformation of $\hat{\mathfrak{h}}$ and $S_o = (s_o)_{*,o}$. The basic properties of ϕ are the following:

PROPOSITION 2.2. (i) *ϕ is an automorphism of the transvection algebra $\mathfrak{t}$.*

(ii) $\phi(\mathfrak{m}) = \mathfrak{m}$.

(iii) $\mathrm{Fix}\phi = \hat{\mathfrak{h}}$.

For the proof of (i) we need the following

LEMMA. (a) $S_o(T_o) = T_o$.

(b) $S_o(R_o) = R_o$.

(c) $R_o(U, V)(S_o) = 0$ *for all* $U, V \in \mathfrak{m}$.

(d) $R_o(U, V) = R_o(S_o U, S_o V)$ *for all* $U, V \in \mathfrak{m}$.

(e) $S_o[U, V] = [U, S_o V]$ *for all* $U \in \hat{\mathfrak{h}}$, $V \in \mathfrak{m}$.

Proof. (a) and (b) are clear. (c) follows from $\nabla S = 0$. (d) is a simple consequence of (b) and (c). It is sufficient to prove (e) for $U = R_o(X, Y)$, $X, Y \in \mathfrak{m}$. In such a case, (e) is nothing but (d).

Proof of Proposition 2.2. (i) We see that ϕ is a vector space isomorphism. Now, it suffices to prove the equality $\phi[U, V] = [\phi U, \phi V]$ in the following three special cases:

1. $U, V \in \hat{\mathfrak{h}}$. Then also $[U, V] \in \hat{\mathfrak{h}}$ and $\phi[U, V] = [U, V] = [\phi U, \phi V]$.
2. $U, V \in \mathfrak{m}$. Then

$$\phi[U, V] = \phi\big(-R_o(U, V), -T_o(U, V)\big) = \big(-R_o(U, V), -S_o(T_o(U, V))\big).$$

By Lemma (d) and (a) we have

$$\begin{aligned}\phi[U, V] &= \big(-R_o(S_o U, S_o V), -T_o(S_o U, S_o V)\big)\\ &= [S_o U, S_o V] = [\phi U, \phi V].\end{aligned}$$

3. $U \in \hat{\mathfrak{h}}$, $V \in \mathfrak{m}$. In this case $\phi[U, V] = S_o[U, V]$ and $[\phi U, \phi V] = [U, S_o V]$. By Lemma (e) we have $\phi[U, V] = [\phi U, \phi V]$.

Assertions (ii) and (iii) of Proposition 2.2 are obvious.

By Proposition 2.2, an *admissible automorphism* of the transvection algebra $\mathfrak{t}$ is an automorphism ϕ of $\mathfrak{t}$ satisfying (ii) and (iii) of that proposition.

THEOREM 2. *There is an order-preserving one-to-one correspondence between the set of all admissible s-structures on (M, ∇) and between the set* *of all admissible automorphisms of the transvection algebra* $\mathfrak{t}$ *of* (M, ∇). *The admissible automorphism* $\phi = \mathrm{id} + (s_o)_{*,o}$ *corresponds to an admissible* s*-structure* $\{s_x\}$.

Proof. By Proposition 1.1, the mapping $\{s_x\} \mapsto \phi = \mathrm{id} + (s_o)_{*,o}$ is injective. Clearly, since the order of the map ϕ is equal to that of $(s_o)_{*,o}$, the above-mentioned correspondence is order preserving. Thus, it remains only to show how to construct an admissible s-structure $\{s_x\}$ from a prescribed admissible automorphism ϕ of $\mathfrak{t}$.

Let G be the connected subgroup of $A(M, \nabla)^0$ corresponding to the algebra $\mathfrak{t}$ and let H be the isotropy subgroup of G at o. The group G acts transitively on M because $\mathfrak{m} \subset \mathfrak{t}$. The isomorphism φ from Proposition 2.1 maps the Lie algebra of H isomorphically onto $\hat{\mathfrak{h}}$. The homogeneous space G/H is reductive and, under the natural identification $M = G/H$, the connection ∇ coincides with the canonical connection on G/H corresponding to the decomposition $\mathfrak{t} = \mathfrak{h} + \mathfrak{m}$. Every admissible automorphism ϕ of the transvection algebra $\mathfrak{t}$ induces an automorphism σ of the group G such that $\sigma_{*,e} = \phi$. A simple homotopy argument shows that the group H is connected, so we have

$$\sigma|H = \mathrm{id}_H. \tag{2}$$

Therefore, there is a well-defined diffeomorphism s of G/H given by

$$s(gH) = \sigma(g)H \quad \text{for all } g \in G. \tag{3}$$

Since $\sigma_{*,e}(m) = m$ and $\sigma(H) = H$, Proposition 3.1 in [1] implies that s is an affine transformation of (M, ∇). It is easy to see that s is a symmetry of (M, ∇) at o. By (2) and (3), s is $\mathrm{ad}(H)$-invariant, so for every $x \in M$ there is a well-defined symmetry s_x at x given by $s_x = g \circ s \circ g^{-1}$, where g is an arbitrary element of G such that $g(o) = x$. We claim that this s-structure $\{s_x\}$ is admissible and that the corresponding admissible automorphism of the transvection algebra is the given ϕ.

Using a local differentiable cross-section in the principal fibre bundle $G(M, H)$ we see that the map $(x, y) \mapsto s_x(y)$ is differentiable. A simple calculation shows that condition (ii) of the definition of an admissible s-structure is also satisfied. Finally, the symmetry tensor field S of $\{s_x\}$ is invariant by G, so S is parallel. This means that the constructed s-structure $\{s_x\}$ is admissible. The equalities

$$\phi = \mathrm{id} + (\phi|\mathfrak{m}) = \mathrm{id} + s_{*,o} = \mathrm{id} + (s_o)_{*,o}$$

complete the proof of Theorem 2.

3. Proof of Theorem 1. In this section we construct a sequence (M_n, ∇_n), $n \geqslant 3$, of g.a.s. spaces satisfying all requirements of Theorem 1. For $n = 3$, the space (M_3, ∇_3) is known from the complete list of g.a.s. spaces of dimension 4 (see [6]).

For every integer $n \geqslant 3$ let us consider the matrix group G_n consisting of the matrices

$$\begin{Vmatrix} e^{u_0} & 0 & \dots & 0 & x_0 \\ 0 & e^{u_1} & \dots & 0 & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{u_{n-1}} & x_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{Vmatrix},$$

where $(x_0, x_1, \dots, x_{n-1}, u_1, \dots, u_{n-1}) \in R^{2n-1}$ is an arbitrary element and $u_0 = -u_1 - \dots - u_{n-1}$. The Lie group G_n is diffeomorphic to the Cartesian space $R^{2n-1}(x_0, x_1, \dots, x_{n-1}, u_1, \dots, u_{n-1})$ in a canonical way. Suppose that the matrices of G_n are identified with the corresponding $(2n-1)$-tuples. Particularly, for the neutral element e of G_n we have $e = (0, \dots, 0)$. Let us consider a 1-parameter subgroup H_n of G_n given by the equalities $x_0 = x_1 = \dots = x_{n-2} = -x_{n-1} = t$, $u_1 = \dots = u_{n-1} = 0$ and the homogeneous space $M_n = G_n/H_n$.

PROPOSITION 3.1. *The manifold M_n is diffeomorphic to R^{2n-2} for all $n \geqslant 3$.*

Proof. The action of H_n on G_n is given by the formula

$$\begin{aligned}(x_0, x_1, \dots, x_{n-1}, u_1, \dots, u_{n-1})\cdot t \\ = (x_0 + te^{u_0}, x_1 + te^{u_1}, \dots, x_{n-2} + te^{u_{n-2}}, x_{n-1} - te^{u_{n-1}}, u_1, \dots, u_n),\end{aligned}$$

where $u_0 = -u_1 - \dots - u_{n-1}$. Therefore, every orbit $g\cdot H_n$, $g \in G$, meets the submanifold G_n' given by $x_0 = 0$ at exactly one point, namely for $t = -x_0 e^{-u_0}$. Thus we have a bijection ψ: $M_n \to G_n'$. Using a differentiable local cross-section in the principal fibre bundle $G_n(M_n, H_n)$, we see easily that the map ψ is differentiable. The inverse map ψ^{-1} is a composition of the inclusion map $G_n' \to G_n$ followed by the canonical projection $G_n \to M_n$, so it is also differentiable. The manifold G_n' is diffeomorphic to R^{2n-2} in a canonical way, which completes the proof of Proposition 3.1.

Note that in formulas of this section we use the following two index sets:

$$i, j, k, \dots \in \{0, 1, \dots, n-1\},$$

$$\alpha, \beta, \gamma, \varkappa, \mu, \dots \in \{1, \dots, n-1\}.$$

A direct calculation shows that the Lie algebra $\mathfrak{g}_n$ of the Lie group G_n has a basis $X_0, X_1, \ldots, X_{n-1}, U_1, \ldots, U_{n-1}$, where

$$X_i = e^{u_i}\frac{\partial}{\partial x_i}, \quad U_a = \frac{\partial}{\partial u_a}. \tag{4}$$

The bracket operation in $\mathfrak{g}_n$ is given by the formulas

$$\begin{gathered}[X_i, X_j] = [U_a, U_\beta] = 0,\\ [X_0, U_a] = X_0, \qquad [X_a, U_\beta] = -\delta_{a\beta}X_a.\end{gathered} \tag{5}$$

In the sequel, we identify the algebra $\mathfrak{g}_n$ with the tangent vector space $T_e(G_n)$.

Let us consider vectors $Y_0, Y_1, \ldots, Y_{n-1}$ defined by

$$\begin{gathered}Y_0 = X_0+X_1+\ldots+X_{n-2}-X_{n-1},\\ Y_a = X_0-X_a \text{ for } a \neq n-1, \quad Y_{n-1} = X_0+X_{n-1}.\end{gathered} \tag{6}$$

The vectors $Y_0, Y_1, \ldots, Y_{n-1}, U_1, \ldots, U_{n-1}$ form a new basis of the algebra $\mathfrak{g}_n$ and the vector Y_0 generates the subalgebra $\mathfrak{h}_n$ corresponding to the subgroup H_n. Formulas (4) and (5) yield

$$\begin{gathered}[Y_i, Y_j] = [U_a, U_\beta] = 0, \quad [Y_0, U_a] = Y_a,\\ [Y_a, U_\beta] = \left(\frac{1+\delta_{a\beta}}{n}\sum_{i=0}^{n-1} Y_i\right) - \delta_{a\beta}Y_a.\end{gathered} \tag{7}$$

For the subspace $\mathfrak{m}_n$ of $\mathfrak{g}_n$ spanned by $Y_1, \ldots, Y_{n-1}, U_1, \ldots, U_{n-1}$ we have $\mathfrak{g}_n = \mathfrak{h}_n+\mathfrak{m}_n$ (vector space direct sum) and $[\mathfrak{h}_n, \mathfrak{m}_n] \subset \mathfrak{m}_n$. Since the group H_n is connected, the homogeneous space $M_n = G_n/H_n$ is reductive. Let ∇_n be the corresponding canonical connection on M_n. By [2], Proposition X.2.6, formulas (7) yield the following expressions for the torsion and curvature of ∇_n. (The vector space $\mathfrak{m}_n$ is identified with the tangent vector space $T_o(M_n)$ in a standard way.)

$$\begin{gathered}T_o(Y_a, Y_\beta) = T_o(U_a, U_\beta) = 0,\\ T_o(Y_a, U_\beta) = -\left(\frac{1+\delta_{a\beta}}{n}\sum_{\varkappa=1}^{n-1} Y_\varkappa\right) + \delta_{a\beta}Y_a,\end{gathered} \tag{8}$$

$$\begin{aligned}R_o(Y_a, Y_\beta)Y_\gamma &= R_o(Y_a, Y_\beta)U_\gamma = R_o(Y_a, U_\beta)Y_\gamma\\ &= R_o(U_a, U_\beta)Y_\gamma = R_o(U_a, U_\beta)U_\gamma = 0,\\ R_o(Y_a, U_\beta)U_\gamma &= -\frac{1+\delta_{a\beta}}{n}U_\gamma.\end{aligned} \tag{9}$$

Now, we calculate the algebra $\mathfrak{a}_n$ of the full affine group $A(M_n, \nabla_n)$. According to Proposition 2.1 we have $\mathfrak{a}_n = \hat{\mathfrak{a}}_n = \mathfrak{m}_n + \hat{\mathfrak{k}}_n$, where

$$\hat{\mathfrak{k}}_n = \{A \in \mathfrak{gl}(\mathfrak{m}_n) | A(R_o) = A(T_o) = 0\}$$

and the multiplication in $\hat{\mathfrak{a}}_n$ is given by (1). By a long but routine calculation it may be shown that the algebra $\hat{\mathfrak{k}}_n$ has a basis consisting of two endomorphisms Y and U of the vector space $\mathfrak{m}_n$ given by

$$Y(U_a) = U(Y_a) = Y_a, \quad Y(Y_a) = U(U_a) = 0.$$

Under the identification $\hat{\mathfrak{a}}_n = \mathfrak{a}_n$ we have $Y = Y_0$. The vectors $Y_0, Y_1, \ldots, Y_{n-1}, U_1, \ldots, U_{n-1}, U$ form a basis of the algebra $\mathfrak{a}_n$. The bracket operation in $\mathfrak{a}_n$ is given by (7) and by

$$[U, Y_i] = Y_i, \quad [U, U_a] = 0. \tag{10}$$

Consequently, the algebra $[\mathfrak{a}_n, \mathfrak{a}_n]$ is spanned by the vectors $Y_0, Y_1, \ldots$ $\ldots, Y_{n-1}$, so $\big[[\mathfrak{a}_n, \mathfrak{a}_n], [\mathfrak{a}_n, \mathfrak{a}_n]\big] = 0$. Hence the algebra $\mathfrak{a}_n$ is solvable. We have proved the following

PROPOSITION 3.2. *The identity component of the group $A(M_n, \nabla_n)$ is solvable for all $n \geqslant 3$.*

In Section 2 we have defined the transvection algebra $\mathfrak{t}$ for every connected and simply connected parallel affine manifold (M, ∇). For $(M, \nabla) = (M_n, \nabla_n)$ we have

PROPOSITION 3.3. *The transvection algebra of (M_n, ∇_n) is the algebra $\mathfrak{g}_n$ for all $n \geqslant 3$.*

Proof. It is sufficient to prove that the algebra $\mathfrak{g}_n$ is generated by the subspace $\mathfrak{m}_n$, but this fact is a simple consequence of (7) and (10).

The set $\mathrm{Aut}_a(\mathfrak{g}_n)$ of all admissible automorphisms of the transvection algebra $\mathfrak{g}_n$ of (M_n, ∇_n) can be characterized in the following manner. Let C_n be the set of all cyclic permutations of the set $\{0, 1, \ldots, n-1\}$. For every $r \in R$ and $\sigma \in C_n$ consider a linear transformation $F(\sigma, r)$ of $\mathfrak{g}_n$ defined by

$$F(\sigma, r)(Y_0) = Y_0,$$

$$\tag{11} \begin{gathered} F(\sigma, r)(Y_a) = \begin{cases} Y_{\sigma(a)} - Y_{\sigma(0)} & \text{if } \sigma(a) \neq 0, \\ -Y_{\sigma(0)} & \text{if } \sigma(a) = 0, \end{cases} \\ F(\sigma, r)(U_a) = \begin{cases} r(Y_{\sigma(a)} - Y_{\sigma(0)}) + U_{\sigma(a)} - U_{\sigma(0)} & \text{if } \sigma(a) \neq 0, \\ -rY_{\sigma(0)} - U_{\sigma(0)} & \text{if } \sigma(a) = 0. \end{cases} \end{gathered}$$

It is easy to check that $F(\sigma, r)$ is an admissible automorphism of $\mathfrak{g}_n$ for every $(\sigma, r) \in C_n \times R$. Hence we have a well-defined injective mapping $F\colon C_n \times R \to \mathrm{Aut}_a(\mathfrak{g}_n)$.

PROPOSITION 3.4. *The map F is bijective.*

Proof. It is enough to prove that F is surjective. Let ϕ be an admissible automorphism of $\mathfrak{g}_n$. By [1], Lemma 6.3, a 1-dimensional vector subspace $\mathfrak{i}$ of $\mathfrak{g}_n$ is an ideal of the algebra $\mathfrak{g}_n$ if and only if $\mathfrak{i}$ is spanned by a vector X_j for some $j \in \{0, 1, \ldots, n-1\}$. Thus we have

$$\phi(X_j) = k_j \cdot X_{\sigma(j)}, \tag{12}$$

where σ is a permutation of the set $\{0, 1, \ldots, n-1\}$ and k_j is a non-zero real number. Since $\mathrm{Fix}\phi = \mathfrak{h}_n$ and the subspace $\mathfrak{h}_n$ is spanned by the vector Y_0, we have $\phi(Y_0) = Y_0$. This result together with (6) and (12) yields

$$\tag{13} \begin{gathered} \phi(Y_0) = Y_0, \\ \phi(Y_a) = \begin{cases} Y_{\sigma(a)} - Y_{\sigma(0)} & \text{if } \sigma(a) \neq 0, \\ -Y_{\sigma(0)} & \text{if } \sigma(a) = 0. \end{cases} \end{gathered}$$

We prove now that σ is a cycle. Let k be the least positive integer for which $\sigma^k(0) = 0$ holds. For every $\delta = 1, \ldots, k$ we put $Y'_\delta = Y_a$, where $a = \sigma^\delta(0)$. The elements of the set $\{Y_1, \ldots, Y_{n-1}\} - \{Y'_1, \ldots, Y'_k\}$ are denoted by $Y'_{k+1}, \ldots, Y'_{n-1}$. By (13), the vector

$$(n-k-1)(Y'_1 + \ldots + Y'_k) - (k+1)(Y'_{k+1} + \ldots + Y'_{n-1})$$

is a fixed element of $\mathfrak{m}_n$ by the transformation ϕ. Since there is no non-zero fixed vector of ϕ in $\mathfrak{m}_n$, we have $k = n-1$; thus σ is a cycle.

It remains only to show that there is a real number r such that $\phi(U_a) = F(\sigma, r)(U_a)$ for all a. Since $\phi(\mathfrak{m}_n) = \mathfrak{m}_n$, there are real numbers $A_a^\varkappa$ and $B_a^\varkappa$ $(a, \varkappa = 1, \ldots, n-1)$ such that

$$\phi(U_a) = \sum_\varkappa A_a^\varkappa Y_\varkappa + \sum_\varkappa B_a^\varkappa U_\varkappa. \tag{14}$$

Applying the transformation ϕ to the identity $0 = [U_a, U_\beta]$ we obtain

$$\begin{aligned} 0 &= \sum_{\varkappa,\mu} (A_a^\varkappa B_\beta^\mu - A_\beta^\varkappa B_a^\mu)[Y_\varkappa, U_\mu] \\ &= \frac{1}{n}\Big(\sum_{\varkappa,\mu} (A_a^\varkappa B_\beta^\mu - A_\beta^\varkappa B_a^\mu) + \sum_\varkappa (A_a^\varkappa B_\beta^\varkappa - A_\beta^\varkappa B_a^\varkappa)\Big)\Big(\sum_i Y_i\Big) - \\ &\quad - \sum_\varkappa (A_a^\varkappa B_\beta^\varkappa - A_\beta^\varkappa B_a^\varkappa) Y_\varkappa. \end{aligned}$$

Since the vectors $Y_1, \ldots, Y_{n-1}, \sum_i Y_i$ are linearly independent, we obtain

$$A_a^{\varkappa} B_\beta^{\varkappa} = A_\beta^{\varkappa} B_a^{\varkappa} \quad \text{for all } a, \beta, \varkappa, \tag{15}$$

$$\sum_{\varkappa} A_a^{\varkappa} \sum_{\mu} B_\beta^{\mu} = \sum_{\varkappa} A_\beta^{\varkappa} \sum_{\mu} B_a^{\mu} \quad \text{for all } a, \beta. \tag{16}$$

Similarly, using the identity $[Y_0, U_a] = Y_a$, we get

$$B_a^{\varkappa} = \begin{cases} +1 & \text{if } \varkappa = \sigma(a), \\ -1 & \text{if } \varkappa = \sigma(0), \\ 0 & \text{otherwise}. \end{cases} \tag{17}$$

From (15) and (17) we have

$$A_a^{\beta} = 0 \quad \text{if } \beta \neq \sigma(a) \text{ or } \beta \neq \sigma(0) \tag{18}$$

and

$$A_a^{\sigma(0)} = A_\beta^{\sigma(0)} \quad \text{for all } a, \beta.$$

Thus, there is a real number r such that

$$A_a^{\sigma(0)} = -r \quad \text{for all } a. \tag{19}$$

Put $\beta = \sigma^{-1}(0)$ and $a \neq \beta$ in (16). By (16)-(19) we obtain

$$A_a^{\sigma(a)} = r \quad \text{for all } a \neq \sigma^{-1}(0). \tag{20}$$

Substituting (17)-(20) in (14) we see that $\phi(U_a) = F(\sigma, r)(U_a)$ for all a, which completes the proof of Proposition 3.4.

Theorem 2 and Proposition 3.4 imply that (M_n, V_n) is a g.a.s. space for all $n \geqslant 3$. It follows from (11) that there are admissible s-structures of finite order on (M_n, V_n) corresponding to admissible automorphisms $F(\sigma, 0)$, $\sigma \in C_n$, and that all of them are of order n. Thus, we have proved the following

PROPOSITION 3.5. *For all $n \geqslant 3$, the affine manifold (M_n, V_n) is a g.a.s. space of order n.*

By Propositions 3.5, 3.1, and 3.2, the g.a.s. space (M_n, V_n) satisfies all requirements of Theorem 1.

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MATEMATICKO-FYZIKÁLNA FAKULTA UK
BRATISLAVA

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