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ON JOINT SPECTRA OF OPERATORS ON A BANACH SPACE ISOMORPHIC TO ITS SQUARE

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It is well known that the joint approximate point spectrum and the joint spectrum coincide on the Banach algebra of all (linear and bounded) operators acting on a complex Hilbert space (see [5]). In this paper we shall give necessary and sufficient conditions that the same result be true in $B(X)$, the Banach algebra of all operators on a complex Banach space X isomorphic to its square. We do not know, however, whether these conditions can be satisfied by a space non-isomorphic to a Hilbert space (cf. Problem 2).

All algebras and linear spaces considered throughout are assumed to be complex. Let A be a Banach algebra with the unit 1. The *left joint spectrum* of an n-tuple $(a_1,\ldots, a_n)$ of elements in A, denoted by $\sigma_l(a_1,\ldots, a_n)$, is defined to be the subset of C^n consisting of those $(\lambda_1,\ldots, \lambda_n)$ for which the system $(a_1-\lambda_1,\ldots, a_n-\lambda_n)$ generates a proper left ideal in A. (We write shortly $a_j-\lambda_j$ for $a_j-\lambda_j 1$.) The *right joint spectrum* $\sigma_r(a_1,\ldots, a_n)$ can be defined in a similar manner. The *joint spectrum*, sometimes called *Harte's spectrum*, is equal to the union of the left and the right joint spectra:

$$\sigma(a_1,\ldots, a_n) = \sigma_l(a_1,\ldots, a_n)\cup\sigma_r(a_1,\ldots, a_n).$$

The *left approximate point spectrum* of an n-tuple $(a_1,\ldots, a_n)$ of elements in A, denoted by $\tau_l(a_1,\ldots, a_n)$, is defined as the set of those n-tuples $(\lambda_1,\ldots, \lambda_n)\in C^n$ for which

$$\inf\Big\{\sum_{j=1}^{n}\|(a_j-\lambda_j)z\|:\ z\in A,\ \|z\| = 1\Big\} = 0.$$

A definition of the *right approximate point spectrum* $\tau_r(a_1,\ldots, a_n)$ is analogous. The *joint approximate point spectrum* is the union of the left and the right approximate point spectra:

$$\tau(a_1,\ldots, a_n) = \tau_l(a_1,\ldots, a_n)\cup\tau_r(a_1,\ldots, a_n).$$

For basic properties of all these spectra see [5].

If $A = B(X)$ and $T_1, \ldots, T_n \in A$, then

$$\tau_1(T_1, \ldots, T_n) = \{(\lambda_1, \ldots, \lambda_n) \in C^n: \inf_{\|x\|=1} \sum_{j=1}^{n} \|(T_j - \lambda_j)x\| = 0\}$$

(τ_1 is often called the *approximate point joint spectrum*), while

$$\tau_r(T_1, \ldots, T_n) = \{(\lambda_1, \ldots, \lambda_n) \in C^n: \sum_{j=1}^{n} (T_j - \lambda_j)X \neq X\}$$

(the so-called *defect spectrum*); see [5], p. 95.

Notice that if $T_1, \ldots, T_n$ are arbitrary elements of A, then

$$\tau_1(T_1, \ldots, T_n) \subset \sigma_1(T_1, \ldots, T_n), \quad \tau_r(T_1, \ldots, T_n) \subset \sigma_r(T_1, \ldots, T_n),$$

and therefore

$$\tau(T_1, \ldots, T_n) \subset \sigma(T_1, \ldots, T_n).$$

It is known (see [5], p. 97) that if X is a Hilbert space, then each of the above inclusions can be replaced by the equality. We shall give necessary and sufficient conditions that the same be true for a Banach space X.

If X is a Banach space with the norm $\| \ \|$, then X^n denotes the Cartesian product of n copies of the space X with the norm of an element $x = (x_1, \ldots, x_n)$ given by the formula

$$\|x\| = \sum_{j=1}^{n} \|x_j\|.$$

Let X and Y be Banach spaces. We write $X \approx Y$ if the spaces X and Y are isomorphic. We say that a Banach space X is *isomorphic to its square* if $X \approx X^2$. A (closed linear) subspace Y of a Banach space X is said to be a *complemented subspace* if there is a projection from X onto Y or, equivalently, if there exists a subspace Z of X such that X is the direct sum of Y and Z, i.e., $X = Y \oplus Z$.

Let X be a Banach space. We shall consider the following two conditions:

(*) *Every subspace Y of X isomorphic to X is complemented in X.*

(**) *Every subspace Y of X such that the quotient space X/Y is isomorphic to X has a complement in X.*

Remarks. 1. It is well known (see, e.g., [4], pp. 30 and 31) that (*) is equivalent to

$$\tau_1(T) = \sigma_1(T) \quad \text{for all } T \in B(X),$$

while (**) coincides with

$$\tau_r(T) = \sigma_r(T) \quad \text{for each } T \in B(X).$$

2. It is standard that

$$\sigma_1(T) = \sigma_r(T^*) \quad \text{and} \quad \sigma_r(T) = \sigma_1(T^*)$$

(T^* is the conjugate operator defined on the conjugate space X^*). Moreover,

$$\tau_l(T) = \tau_r(T^*) \quad \text{and} \quad \tau_r(T) = \tau_l(T^*)$$

(see [8] and [9]). These facts imply that:

(a) Conditions (*) and (**) are equivalent in the class of reflexive Banach spaces.

(b) If (*) (respectively, (**)) is not satisfied in a Banach space X, then (**) (respectively, (*)) does not hold in X^*.

(*) is satisfied in c_0, l^2, l^∞ (see [6]) but is not satisfied in l^p, $1 < p < \infty$, $p \neq 2$ (see [1] and [7]), and in l^1 (see [2] or [3]). (**) holds in l^1 and l^2 but c_0, l^∞, and l^p, $1 < p < \infty$, $p \neq 2$, do not have this property.

PROPOSITION. *Let X be a Banach space isomorphic to its square.*

(i) *If X satisfies* (*), *then*

$$\tau_l(T_1, \ldots, T_n) = \sigma_l(T_1, \ldots, T_n)$$

for an arbitrary n-tuple $(T_1, \ldots, T_n)$ of operators in $B(X)$.

(ii) *If X satisfies* (**), *then*

$$\tau_r(T_1, \ldots, T_n) = \sigma_r(T_1, \ldots, T_n)$$

for an arbitrary n-tuple $(T_1, \ldots, T_n)$ of operators in $B(X)$.

Proof. (i) It is enough to show that

$$(0, \ldots, 0) \notin \tau_l(T_1, \ldots, T_n)$$

implies

$$(0, \ldots, 0) \notin \sigma_l(T_1, \ldots, T_n) \quad (n \geqslant 2).$$

Assuming $(0, \ldots, 0) \notin \tau_l(T_1, \ldots, T_n)$ we get $\delta > 0$ such that

$$\|T_1 x\| + \ldots + \|T_n x\| \geqslant \delta \|x\| \quad \text{for all } x \in X.$$

Define the operator T: $X \to X^n$ by the formula

$$Tx = (T_1 x, \ldots, T_n x).$$

Then $\|Tx\| \geqslant \delta \|x\|$ for every $x \in X$. Hence T is bounded from below, which means that it is an isomorphism into. Moreover, $X \approx X^n$. Let Φ be an isomorphism from X^n onto X. Then the mapping $\Phi \circ T$: $X \to X$ is an isomorphism into. Put $(\Phi \circ T)X = Y$. It is a closed subspace of X isomorphic to X, and hence complemented in X. Let P: $X \to Y$ be a projection and let

$$S = (\Phi \circ T)^{-1}: Y \to X.$$

We have $S \circ P \circ \Phi \circ T = I$ (the identity operator on X). If J_k: $X \to X^n$ is the operator which sends x onto $(0, \ldots, 0, x, 0, \ldots, 0)$, where x is at the k-th place, then defining

$$S_k = S \circ P \circ \Phi \circ J_k \quad (k = 1, \ldots, n),$$

we shall get elements of $B(X)$ such that

$$\sum_{k=1}^{n} S_k T_k = \sum_{k=1}^{n} S \circ P \circ \Phi \circ J_k \circ T = S \circ P \circ \Phi \circ T = I.$$

But this means that $(0,\ldots, 0) \notin \sigma_l(T_1,\ldots, T_n)$.

(ii) As before it is enough to show that

$$(0,\ldots, 0) \notin \tau_r(T_1,\ldots, T_n)$$

implies

$$(0,\ldots, 0) \notin \sigma_r(T_1,\ldots, T_n).$$

Let $(0,\ldots, 0) \notin \tau_r(T_1,\ldots, T_n)$. Then

$$\sum_{j=1}^{n} T_j X = X.$$

Define the operator T: $X^n \to X$ by the formula

$$T(x_1,\ldots, x_n) = \sum_{j=1}^{n} T_j x_j.$$

Then T is onto. Thus, if Ψ: $X \to X^n$ is an isomorphism, then $T \circ \Psi$ maps X onto itself. Let $Y = \ker(T \circ \Psi)$. Then Y is a closed subspace of X and $X/Y \approx X$. Y is complemented in X by our assumption. Therefore there exists a closed subspace Z of X such that $X = Y \oplus Z$. By the open mapping theorem, $T \circ \Psi$ is open, and so $(T \circ \Psi)|_Z$ is open. Moreover, $(T \circ \Psi)|_Z$ is one-to-one, and hence it is an isomorphism. Let S: $X \to Z$ be the inverse of $(T \circ \Psi)|_Z$. Thus we have

$$(T \circ \Psi)|_Z \circ S = I.$$

Let P_j: $Z^n \to Z$ be the projection onto the j-th coordinate, i.e.,

$$P_j(z_1,\ldots, z_n) = z_j \quad (j = 1,\ldots, n),$$

and let

$$S_j = P_j \circ (\Psi|_Z) \circ S: \ X \to Z.$$

Then we get

$$\sum_{j=1}^{n} T_j S_j = T \circ (\Psi|_Z) \circ S = (T \circ \Psi)|_Z \circ S = I,$$

which means that $(0,\ldots, 0) \notin \sigma_r(T_1,\ldots, T_n)$.

Remark. We do not know if the assumption $X \approx X^2$ in the Proposition can be omitted (cf. Problem 1).

COROLLARY. *If a Banach space X isomorphic to its square satisfies both* $(*)$ *and* $(**)$, *then*

$$\tau(T_1,\ldots,T_n)=\sigma(T_1,\ldots,T_n)$$

for an arbitrary n-tuple $(T_1,\ldots,T_n)$ *of operators in* $B(X)$.

Now we shall show that the converse is also true. More precisely, we prove the following

THEOREM. *Let X be a Banach space isomorphic to its square. If* $\tau(T,S)=\sigma(T,S)$ *for arbitrary* $T, S\in B(X)$, *then X satisfies both* $(*)$ *and* $(**)$.

Proof. Since $X\approx X^2$, there is a one-to-one correspondence between operators on X and elements of $B(X^2)$. Namely, if $\Psi: X\to X^2$ and $\Phi: X^2\to X$ are isomorphisms such that $\Phi\circ\Psi=I$ and $\Psi\circ\Phi=I$, then the function

$$T\mapsto\tilde{T}=\Psi\circ T\circ\Phi$$

maps isomorphically $B(X)$ onto $B(X^2)$. Moreover, we have

$$\sigma_l(T_1,\ldots,T_n)=\sigma_l(\tilde{T}_1,\ldots,\tilde{T}_n),\qquad \sigma_r(T_1,\ldots,T_n)=\sigma_r(\tilde{T}_1,\ldots,\tilde{T}_n),$$
$$\tau_l(T_1,\ldots,T_n)=\tau_l(\tilde{T}_1,\ldots,\tilde{T}_n),\qquad \tau_r(T_1,\ldots,T_n)=\tau_r(\tilde{T}_1,\ldots,\tilde{T}_n).$$

It will be convenient to represent operators on X^2 in the matrix form, i.e., if $T\in B(X^2)$, then T can be regarded as a matrix

$$\begin{bmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$$

with $T_{jk}\in B(X)$, $j, k=1,2$.

Now, suppose that $(*)$ is not satisfied, i.e., there exists an operator $T\in B(X)$ such that $0\in\sigma_l(T)$ but $0\notin\tau_l(T)$. Take the following two operators on $B(X^2)$:

$$S=\begin{bmatrix} T & 0\\ 0 & I\end{bmatrix}\quad\text{and}\quad R=\begin{bmatrix} 0 & -\lambda\\ 0 & I\end{bmatrix},$$

where λ is an arbitrary fixed complex number outside the spectrum of T.

We claim that $(0,0)\in\sigma(S,R)$ but $(0,0)\notin\tau(S,R)$. To see this notice first that $(0,0)\in\sigma_l(S,R)$. For otherwise there would exist operators $U, V\in B(X^2)$ such that $US+VR=I$. Computing elements in the first row and the first column we would get $U_{11}T=I$, which would be impossible.

Now observe that $(0,0)\notin\tau_l(S,R)$. Otherwise, we would get a sequence

$$x_n=(x_1^{(n)},x_2^{(n)})\in X^2$$

such that

$$\|x_n\|=\|x_1^{(n)}\|+\|x_2^{(n)}\|=1\quad\text{for all } n$$

and

$$\|Sx_n\|\to 0,\quad \|Rx_n\|\to 0\quad\text{as } n\to\infty.$$

But
$$\|Sx_n\| = \|Tx_1^{(n)}\| + \|x_2^{(n)}\| \quad \text{and} \quad \|Rx_n\| = (|\lambda|+1)\|x_2^{(n)}\|.$$
This would imply
$$\|x_2^{(n)}\| \to 0, \quad \|x_1^{(n)}\| \to 1, \quad \|Tx_1^{(n)}\| \to 0 \quad \text{as } n \to \infty.$$
Finally, we would obtain $0 \in \tau_1(T)$, a contradiction.

Further, $(0, 0) \notin \tau_r(S, R)$. To see this take an arbitrary $x = (x_1, x_2) \in X^2$. Then there exists $y \in X$ such that $(T-\lambda)y = x_1$. Thus
$$S(y, x_2-y)+R(z, y) = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} y \\ x_2-y \end{bmatrix} + \begin{bmatrix} 0 & -\lambda \\ 0 & I \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} (T-\lambda)y \\ x_2 \end{bmatrix} = x$$
($z \in X$ is arbitrary).

Hence we get
$$(0, 0) \in \sigma_1(S, R) \subset \sigma(S, R)$$
and
$$(0, 0) \notin \tau_1(S, R) \cup \tau_r(S, R) = \tau(S, R).$$
Since $\bar{S} = \Phi \circ S \circ \Psi$ and $\bar{R} = \Phi \circ R \circ \Psi$ are elements of $B(X)$ and
$$\sigma(\bar{S}, \bar{R}) = \sigma(S, R), \quad \tau(\bar{S}, \bar{R}) = \tau(S, R),$$
we have $\tau(\bar{S}, \bar{R}) \neq \sigma(\bar{S}, \bar{R})$.

If the space X does not satisfy (**), then the proof goes similarly, i.e., it is enough to take the same operator S and
$$R = \begin{bmatrix} 0 & 0 \\ \lambda & I \end{bmatrix}, \quad \lambda \neq 0.$$

COROLLARY. *Let a Banach space X be isomorphic to its square. If $\tau(T, S) = \sigma(T, S)$ for arbitrary $T, S \in B(X)$, then*
$$\tau(T_1, \ldots, T_n) = \sigma(T_1, \ldots, T_n)$$
for every finite subset $\{T_1, \ldots, T_n\}$ of $B(X)$.

Concluding remarks. 1. A similar theorem can be proved for a complex Banach space isomorphic to its n-th power ($n > 2$).

2. All the results above have been obtained under the assumption that the Banach space in question is isomorphic to its square. We do not know if this assumption can be dropped. Therefore we pose

PROBLEM 1 (**P 1370**). Do the Theorem and the Proposition hold true without the assumption $X \approx X^2$?

We do not know any example of a Banach space X non-isomorphic to a Hilbert space which satisfies both (*) and (**). Hence we formulate the following

PROBLEM 2 (**P 1371**). Is it true that any such space must be isomorphic to a Hilbert space?

It would be also interesting to know an answer to the following

PROBLEM 3 (**P 1372**). Is it possible to consider only commuting systems of operators in the assumptions of the Theorem (maybe with the number of operators greater than 2)?

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