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CHARACTERIZATIONS OF GENERALIZED PARACOMPACT SPACES

BY

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1. Introduction. The notion of paracompactness and many of its generalizations have been extensively studied (see [2], [4], [5] and [13]), and they have been shown to play important roles in metrization theorems, the normal Moore space conjecture and other related problems.

In 1969 Worrell and Wicke [23] introduced the concept of a θ-refinable space, and a technique of Michael [14] showed that collectionwise normal, θ-refinable spaces were paracompact. Since the notion of θ-refinability was defined in terms of a sequence of open refinements with a special property, it was natural to ask whether other generalized paracompact spaces had such an equivalence. In [6] Burke gave such a result for the class of subparacompact spaces.

THEOREM 1.1 (Burke). *A space X is subparacompact iff every open cover $\mathscr{G}$ of X has a refinement $\bigcup_{i=1}^{\infty}\mathscr{G}_i$ such that*

(i) *each $\mathscr{G}_i$ is an open cover of X, and*

(ii) *for each $x \in X$ there exists an $n(x)$ such that* $\operatorname{ord}(x, \mathscr{G}_{n(x)}) = 1$.

In 1973 Boyte [4] obtained a similar result for paracompactness. In this paper we provide analogous θ-type characterizations for various generalized paracompact spaces. In Section 2 we use the notion of θ-expansions as discussed in [19] to give a simple proof of Boyte's result, Theorem 2.3 below. In Section 3 similar characterizations are given for metacompactness, and the properties of screenability, sequential mesocompactness, and mesocompactness are characterized in Section 4. Finally, in Section 5 several results are obtained illustrating the usefulness of the obtained characterizations.

All spaces are T_1 unless otherwise stated, N denotes the natural numbers and m represents an infinite cardinal. The following notions concerning expandability are included for the benefit of the reader. A detailed study is found in [11] and [20].

Definition 1.1. A space X is called (*discretely*) *m-expandable* if for every (discrete) locally finite collection $\{F_a\colon a \in A\}$ of closed subsets of X, with $|A| \leqslant m$, there exists a locally finite open collection $\{G_a\colon a \in A\}$ such that $F_a \subseteq G_a$ for each $a \in A$. If the open collection $\{G_a\colon a \in A\}$ is only required to be point finite, then the word "almost" preceds the above-mentioned terminology.

THEOREM 1.2. (1) *A space X is expandable iff X is discretely expandable and countably paracompact.*

(2) *A space X is almost expandable iff X is almost discretely expandable and countably metacompact.*

2. θ-characterizations for paracompactness.

Definition 2.1. A space X is said to satisfy *property* (m, P) provided every open cover $\mathscr{G}$ of X, consisting of at most m members, has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ such that

(i) each $\mathscr{G}_i$ is an open cover of X, and

(ii) for each $x \in X$, there exist a neighborhood $U(x)$ of x and an integer $n(x)$ such that $U(x)$ intersects only finitely many members of $\mathscr{G}_{n(x)}$.

If every open cover of X has such a refinement, then X is said to have *property P*.

In [3] and [4] Boyte obtained characterizations for countable paracompactness and paracompactness using the above-mentioned notion. The proofs were somewhat complicated however, and in the later case it was necessary to first show that the space was normal. Here we furnish alternate proofs using the notion of θ-expandability. For the benefit of the reader we include this definition and a result found in [19].

Definition 2.2. A space X is called *m-θ-expandable* if for every locally finite collection $\mathscr{F} = \{F_a\colon a \in A\}$, with $|A| \leqslant m$, there exist open collections $\mathscr{G}_i = \{G(a, i)\colon a \in A\}$ for $i = 1, 2, \ldots$ such that

(i) $F_a \subseteq G(a, i)$ for each $a \in A$ and each i, and

(ii) for each $x \in X$ there exists an integer $n(x)$ such that $\mathscr{G}_{n(x)}$ is locally finite at x.

THEOREM 2.1. (1) *A space X is countably paracompact iff X is $\aleph_0$-θ-expandable.*

(2) *A space X is m-paracompact iff X is m-θ-refinable and m-θ-expandable.*

Remark. The notion of θ-expandability is a generalization of the notion of expandability as discussed in [11] and [20].

THEOREM 2.2. *If X satisfies property (m, P), then X is m-θ-expandable.*

Proof. Let $\mathscr{F} = \{F_a\colon a \in A\}$ be a locally finite collection of closed subsets of X with $|A| \leqslant m$. For the collection Γ of all finite subsets of A, we write

$$G(B) = X - \bigcup_{a \notin B} F_a \quad \text{for each } B \in \Gamma.$$

Hence $\mathscr{G} = \{G(B)\colon B \in \Gamma\}$ is an open cover of X, consisting of at most m members, such that each $G(B)$ intersects only finitely many members of $\mathscr{F}$. Let $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ be a refinement of $\mathscr{G}$ satisfying properties (i) and (ii) of Definition 2.1.

Write $U(a, n) = \mathrm{St}(F_a, \mathscr{G}_n)$ for each $a \in A$ and each n. Clearly, $F_a \subseteq U(a, n)$ for each $a \in A$ and each n; and, for each $x \in X$, $\{G(a, n)\colon a \in A\}$ is locally finite at x for some n. Therefore, X is m-θ-expandable.

THEOREM 2.3. *A space X is m-paracompact iff X satisfies property (m, P).*

Proof. Suppose X satisfies property (m, P). Then, by Theorem 2.2, X is m-θ-refinable and m-θ-expandable. Therefore, X is m-paracompact by Theorem 2.1.

COROLLARY 2.1. (1) *A space X is paracompact iff X satisfies property P.*

(2) *A space X is countably paracompact iff X satisfies property $(\aleph_0, P)$.*

We now observe that another characterization for paracompactness can be obtained using a finite collection of covers. Analogous results are later given for the other properties.

Definition 2.3. A space X is said to satisfy *property* $[m, P]$ if every open cover $\mathscr{G}$ of X, consisting of at most m members, has a refinement $\bigcup_{i=1}^{k} \mathscr{G}_i$ such that

(i) each $\mathscr{G}_i$ is an open cover of X, and

(ii) for each $x \in X$, there exist a neighborhood $U(x)$ of x and an integer $n(x)$, $1 \leqslant n(x) \leqslant k$, such that $U(x)$ intersects only finitely many members of $\mathscr{G}_{n(x)}$.

THEOREM 2.4. *A space X is m-paracompact iff X satisfies property* $[m, P]$.

Proof. The proof follows from Theorem 2.3 and the fact that property $[m, P]$ implies property (m, P).

3. θ-characterizations for metacompactness. We now show that metacompactness has characterizations analogous to those obtained for paracompactness in the preceding section.

Definition 3.1. A space X is said to have *property* (m, M) provided every open cover $\mathscr{G}$ of X, consisting of at most m members, has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ such that

(i) each $\mathscr{G}_i$ is an open cover of X, and

(ii) for each $x \in X$, there exist a neighborhood $U(x)$ of x and an integer $n(x)$ such that $\operatorname{ord}(y, \mathscr{G}_{n(x)}) < \infty$ for $y \in U(x)$.

As before, X has *property* M if X has property (m, M) for all cardinals m.

THEOREM 3.1. *A space X is countably metacompact iff X satisfies property* $(\aleph_0, M)$.

Proof. The proof follows from the fact that if X satisfies property $(\aleph_0, M)$, then it is $\aleph_0$-θ-refinable. Gittings [8] has shown that $\aleph_0$-θ-refinability is equivalent to countable metacompactness.

THEOREM 3.2. *A space X is m-metacompact iff X satisfies property* (m, M).

Proof. Suppose X satisfies property (m, M). Then X is m-θ-refinable and countably metacompact. By Theorem 4.3 of [20], it is sufficient to show that X is almost m-discretely expandable. Let $\mathscr{F} = \{F_a\colon a \in A\}$ be a discrete collection of closed subsets of X with $|A| \leqslant m$. For each $a \in A$, let G_a be an open set containing F_a such that $G_a \cap F_\beta = \emptyset$ for $\beta \neq a$. Then

$$\mathscr{G} = \{G_a\colon a \in A\} \cup \{X - \bigcup_{a \in A} F_a\}$$

is an open cover of X. Since X has property (m, M), $\mathscr{G}$ has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ satisfying properties (i) and (ii) of Definition 3.1. Write

$$G(a, 1) = \operatorname{St}(F_a, \mathscr{G}_1) \cap G_a \quad \text{for each } a \in A$$

and

$$G(a, n) = \operatorname{St}(F_a, \mathscr{G}_n) \cap G(a, n-1) \quad \text{for } n > 1,$$

so that

$$F_a \subseteq G(a, n+1) \subseteq G(a, n) \subseteq G_a \quad \text{for each } a \in A \text{ and each } n.$$

Let V_i be the set of all points x such that each x has a neighborhood $U(x)$ with $\operatorname{ord}(y, \mathscr{G}_i) < \infty$ for $y \in U(x)$.

It is easy to see that $\mathscr{V} = \{V_i\}_{i=1}^{\infty}$ is a countable open cover of X, and hence has a point finite open refinement $\mathscr{W} = \{W_i\}_{i=1}^{\infty}$ such that $W_i \subseteq V_i$ for each i.

Write $H(a, i) = W_i \cap G(a, i)$ for each $a \in A$ and each i, and $\mathscr{H} = \{H(a, i)\colon a \in A;\ i = 1, 2, \ldots\}$.

ASSERTION 1. $\mathscr{H}$ is an open cover of $\bigcup_{a \in A} F_a$. If $x \in \bigcup_{a \in A} F_a$, then $x \in F_{a_0}$ for some $a_0 \in A$ and $x \in G(a_0, n)$ for all n. But $x \in W_j$ for some j, and hence

$$x \in W_j \cap G(a_0, j) = H(a_0, j).$$

ASSERTION 2. $\mathscr{H}$ is point finite. If $x \in X$, then x belongs to only finitely many members of $\mathscr{W}$. Also $x \in W_i \subseteq V_i$ implies that x belongs to only finitely many members of $\{G(a, i): a \in A\}$. Hence x belongs to only finitely many members of $\mathscr{H}$. Therefore, X is almost m-discretely expandable by Lemma 3.6 of [20] and the proof is complete.

COROLLARY 3.1. *A space X is metacompact iff X satisfies property M.*

Definition 3.2. A space X is said to satisfy *property* $[m, M]$ provided every open cover $\mathscr{G}$ of X, consisting of at most m members, has a refinement $\bigcup_{i=1}^{k} \mathscr{G}_i$ such that each $\mathscr{G}_i$ is an open cover of X and, for each $x \in X$, there exists an integer $n(x)$, $1 \leqslant n(x) \leqslant k$, such that $\operatorname{ord}(x, \mathscr{G}_{n(x)}) < \infty$.

Remark. If a space X satisfies property $[m, M]$ for all m, we could appropriately say that X is *finitely θ-refinable.* The next result actually shows that this notion is equivalent to metacompactness.

THEOREM 3.3. *A space X is m-metacompact iff X satisfies property $[m, M]$.*

Proof. Let X satisfy property $[m, M]$. Then X is m-θ-refinable, and hence countably metacompact. Therefore, by Theorem 2.8 of [20], it is sufficient to show that X is almost m-discretely expandable. Let $\mathscr{F} = \{F_a: a \in A\}$ be a discrete collection of closed subsets of X with $|A| \leqslant m$. For each $a \in A$ let G_a be an open set containing F_a such that $G_a \cap F_\beta = \emptyset$ for $\beta \neq a$. Since X satisfies property $[m, M]$, we infer that

$$\mathscr{G} = \{G_a: a \in A\} \cup \{X - \bigcup_{a \in A} F_a\}$$

has a refinement $\bigcup_{i=1}^{k} \mathscr{G}_i$ satisfying (i) and (ii) of Definition 3.1. As before, let $G(a, i) = \operatorname{St}(F_a, \mathscr{G}_i)$ for each i, $1 \leqslant i \leqslant k$.

Write

$$U_a = \bigcap_{i=1}^{k} G(a, i) \quad \text{for each } a \in A.$$

Clearly, $F_a \subseteq U_a$ for each $a \in A$ and it is easy to show that $\{U_a: a \in A\}$ is point finite. Therefore, X is almost m-discretely expandable.

COROLLARY 3.2. (1) *A space X is countably metacompact iff X satisfies property* $[\aleph_0, M]$.

(2) *A space X is metacompact iff X is finitely θ-refinable.*

4. Screenability, sequential mesocompactness and mesocompactness. In this section we give characterizations for the above-described classes of spaces which are analogous to those in the previous sections. For the sake of simplicity, and the reader, we omit the cardinality conditions on the open covers. We also omit the proofs for the special case where $m = \aleph_0$, since these proofs are similar to those already presented.

Definition 4.1. A space X is said to have *property* S provided every open cover $\mathcal{G}$ of X has a refinement $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ such that

(i) each $\mathcal{G}_i$ is an open cover of X, and

(ii) for each $x \in X$, there exist a neighborhood $U(x)$ of x and an integer $n(x)$ such that

$$\operatorname{ord}(y, \mathcal{G}_{n(x)}) \leqslant \operatorname{ord}(x, \mathcal{G}_{n(x)}) < \infty \quad \text{for } y \in U(x).$$

THEOREM 4.1. (1) *If X satisfies property S, then X is screenable.*

(2) *If X is normal, countably paracompact and screenable, then X satisfies property S.*

Proof. (1) Suppose that X satisfies property S and $\mathcal{G}$ is an open cover of X. Then $\mathcal{G}$ has a refinement $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying conditions (i) and (ii) of Definition 4.1.

Write

$$U(i, j) = \{x\colon x \text{ has a neighborhood } U(x) \text{ such that } \operatorname{ord}(y, \mathcal{G}_i) \leqslant \operatorname{ord}(x, \mathcal{G}_i) \leqslant j \text{ for } y \in U(x)\} \quad \text{for each } i \text{ and each } j.$$

Then $\{V(i,j)\}_{i,j=1}^{\infty}$ is an open cover of X. Now write $\mathcal{G}_i = \{G(a, i)\colon a \in A_i\}$ for each i, and let $\Gamma(i,j) = \{B \subseteq A_i\colon |B| = j\}$ for each i and j. Let

$$W(B) = U(i,j) \cap \Big[\bigcap_{a \in B} G(a, i)\Big] \quad \text{for each } B \in \Gamma(i,j)$$

and let

$$\mathcal{W}(i,j) = \{W(B)\colon B \in \Gamma(i,j)\}.$$

It is easy to show that

$$\mathcal{W} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{W}(i,j)$$

is a σ-disjoint open refinement of $\mathcal{G}$ so that X is screenable.

(2) Let X be normal, countably paracompact and screenable, and let $\mathcal{G}$ be an open cover of X. Then $\mathcal{G}$ has a σ-disjoint open refinement $\bigcup_{i=1}^{\infty} \mathcal{U}_i$. Define $U_i = \bigcup \{U\colon U \in \mathcal{U}_i\}$ so that $\{U_i\}_{i=1}^{\infty}$ covers X. Since X is countably paracompact and normal, there exist locally finite open covers $\mathcal{V} = \{V_i\}_{i=1}^{\infty}$ and $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ such that $V_i \subseteq \bar{V}_i \subseteq W_i \subseteq U_i$ for each i. Write

$$\mathcal{F}_k = \{F \subseteq N\colon |F| = k\}$$

and

$$\mathcal{H}_k(F) = \{H(k, F, i)\}_{i=1}^{\infty} = \{W_i\colon i \notin F\} \cup \{V_j\colon j \in F\} \quad \text{for each } F \in \mathcal{F}_k.$$

Clearly, for each k and each $F \in \mathscr{F}_k$, $\mathscr{H}_i(F)$ covers X. Let

$$\mathscr{B} = \{B_i = \text{bdry}(W_i)\}_{i=1}^{\infty}.$$

If $\text{ord}(x, \mathscr{B}) = k$ for $x \in X$, there exists a neighborhood $U(x)$ of x such that

$$x \in H \in \mathscr{H}_k(F) \text{ iff } U(x) \cap H \neq \emptyset \quad \text{for some } F \in \mathscr{F}_k.$$

Also, if $y \in U(x)$, then $\text{ord}(y, \mathscr{H}_k(F)) \leqslant \text{ord}(x, \mathscr{H}_k(F))$. Now let

$$\mathscr{G}_k(F) = \{H(k, F, i) \cap \mathscr{U}_i\}_{i=1}^{\infty}.$$

Then $\bigcup_{k=1}^{\infty} \{\mathscr{G}_k(F) \colon F \in \mathscr{F}_k\}$ is a collection of open covers satisfying (i) and (ii) of Definition 4.1. Hence X satisfies property S.

Definition 4.2. A space X is said to have *property SM* provided every open cover $\mathscr{G}$ of X has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ such that

(i) each $\mathscr{G}_i$ is an open cover of X, and

(ii) for each $x \in X$ there exist a neighborhood $U(x)$ of x and an integer $n(x)$ such that if $\{x_j\}_{j=1}^{\infty}$ converges in $U(x)$, then $\{x_j\}_{j=1}^{\infty}$ intersects only finitely many members of $\mathscr{G}_{n(x)}$.

THEOREM 4.2. *Let X be countably paracompact. Then X is sequentially mesocompact iff it has property SM.*

Proof. Suppose X satisfies property SM. Since X is θ-refinable and countably paracompact, we need only to show that if $\{F_a \colon a \in A\}$ is a discrete collection of closed subsets of X, then there exists an open collection $\mathscr{H} = \{H_a \colon a \in A\}$ with $F_a \subseteq H_a$ such that each convergent sequence in X intersects only finitely many members of $\mathscr{H}$. (Boone [1] refers to this notion as property (ω).) Choose G_a to be an open set containing F_a such that $G_a \cap F_\beta = \emptyset$ for $\beta \neq a$. Again

$$\mathscr{G} = \{G_a \colon a \in A\} \cup \{X - \bigcup_{a \in A} F_a\}$$

is an open cover of X and has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ satisfying (i) and (ii) of Definition 4.2.

Let

$$G(a, 1) = G_a \cap \text{St}(F_a, \mathscr{G}_1) \quad \text{for each } a \in A$$

and let

$$G(a, n) = G(a, n-1) \cap \text{St}(F_a, \mathscr{G}_n) \quad \text{for each } n > 1.$$

If $U_i = \{x \colon x$ satisfies (ii) of Definition 4.2 for $n(x) = i\}$, then $\{U_i\}_{i=1}^{\infty}$ is a monotone increasing open cover of X. Since X is countably paracompact, there exists a locally finite open refinement $\{V_i\}_{i=1}^{\infty}$ such that $V_i \subseteq \overline{V}_i \subseteq U_i$.

Now let $H(a, n) = V_n \cap G(a, n)$ for each $a \in A$ and each n, and let $\mathscr{H} = \{H(a, n): a \in A;\ n = 1, 2, \ldots\}$. Clearly, $\mathscr{H}$ is an open cover of $\bigcup_{a \in A} F_a$. We assert that if $\{x_j\}_{j=1}^{\infty}$ converges in X, then $\{x_j\}_{j=1}^{\infty}$ intersects only finitely many members of $\mathscr{H}$, and hence X has property (ω). Suppose $\{x_j\} \to x$. There exists a neighborhood U_0 of x such that U_0 intersects only $V_{j_1}, V_{j_2}, \ldots, V_{j_m}$. Also we may assume that $U_0 \cap V_{j_k} \neq \emptyset$ iff $x \in \bar{V}_{j_k}$ for $1 \leqslant k \leqslant m$. Now, for each k, $1 \leqslant k \leqslant m$, x has a neighborhood U_{j_k} such that $\{x_j\}_{j=1}^{\infty}$ intersects only finitely many members of $\mathscr{G}_{j_k}$. Therefore, if

$$U(x) = U_0 \cap \Big[\bigcap_{k=1}^{m} U_{j_k}\Big],$$

then every convergent sequence in $U(x)$ intersects only finitely many members of $\mathscr{H}$. Thus $\{x_j\}_{j=1}^{\infty}$ intersects only finitely many members of $\mathscr{H}$, and hence X has property (ω).

Definition 4.3. A space X is said to have *property MES* provided every open cover $\mathscr{G}$ of X has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ such that

(i) each $\mathscr{G}_i$ is an open cover of X, and

(ii) for each $x \in X$, there exist a neighborhood $U(x)$ of x and an integer $n(x)$ such that if K is compact and $K \subseteq U(x)$, then K intersects only finitely many members of $\mathscr{G}_{n(x)}$.

THEOREM 4.3. *A countably paracompact space X is mesocompact iff X satisfies property MES.*

The proof of this theorem is essentially the same as that for Theorem 4.2, and hence is omitted.

5. Applications. We now show how the previous characterizations can be used to obtain various results in a simple manner.

THEOREM 5.1. *Every F_σ-subset of a metacompact space X is metacompact.*

Proof. Let

$$F = \bigcup_{i=1}^{\infty} F_i$$

be an F_σ-subset of X, and $\mathscr{G}$ an open cover of F. Let $\mathscr{G}^*$ be a collection of open sets in X such that $\mathscr{G}^* | F = \mathscr{G}$. For each i, $\mathscr{G}_i = \mathscr{G}^* \cup \{X - F_i\}$ covers X, and hence has a refinement $\bigcup_{j=1}^{\infty} \mathscr{G}_{ij}$ satisfying (i) and (ii) of Definition 3.1. Clearly, $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathscr{G}_{ij} | F$ is the desired refinement of $\mathscr{G}$. Hence F is metacompact.

Remark. Analogous results for the other generalized notions of paracompactness also follow.

THEOREM 5.2. *The Countable Sum Theorem is true for θ-refinability.*

Proof. Suppose

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where X_i is a closed θ-refinable subspace of X. Let $\mathscr{G}$ be an open cover of X. Then $\mathscr{G}|X_i$ has a refinement $\bigcup_{j=1}^{\infty} \mathscr{G}_{ij}$ satisfying the definition for θ-refinability. Let $\bigcup_{j=1}^{\infty} \mathscr{G}_{ij}^*$ be a collection of open subsets of X, each member of which is contained in some member of $\mathscr{G}$ and such that

$$\bigcup_{j=1}^{\infty} \mathscr{G}_{ij}^*|X_i = \bigcup_{j=1}^{\infty} \mathscr{G}_{ij}.$$

Write $\mathscr{V}_{ij} = \mathscr{G}_{ij}^* \cup \{G - X_i\colon G \in \mathscr{G}\}$ for each i and each j. Then $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathscr{V}_{ij}$ refines $\mathscr{G}$ and satisfies the property that, for $x \in X$, there exists some $\mathscr{V}_{ij}$ such that $\mathrm{ord}(x, \mathscr{V}_{ij}) < \infty$. Hence X is θ-refinable.

COROLLARY 5.1. *The Countable Sum Theorem is true for countable metacompactness.*

This follows since countable metacompactness is equivalent to $\aleph_0$-θ-refinability.

In [20] the authors established the following

THEOREM 5.3. *The Locally Finite Sum Theorem is true for almost expandable spaces.*

THEOREM 5.4 (Bounded Locally Finite Sum Theorem). *Let*

$$X = \bigcup_{a \in A} F_a,$$

$\mathscr{F} = \{F_a\colon a \in A\}$ being a collection of subsets of X such that, for some integer k, each $x \in X$ has a neighborhood which intersects at most k members of $\mathscr{F}$. If each F_a is metacompact, then X is metacompact.

Proof. The proof is by induction on k. For $k = 1$, $\mathscr{F}$ is discrete, and hence it is immediate that X is metacompact.

Suppose the theorem is true for all $k \leqslant n$, and let $\mathscr{F} = \{F_a\colon a \in A\}$ be a collection of closed metacompact subsets of X such that each $x \in X$ has a neighborhood which intersects at most $n+1$ members of $\mathscr{F}$. Write

$$\Gamma = \{B \subseteq A\colon |B| = n+1\} \quad \text{and} \quad H(B) = \bigcap_{a \in B} F_a \quad \text{for each } B \in \Gamma.$$

It is easy to see that $\mathscr{H} = \{H(B)\colon B \in \Gamma\}$ is discrete so that $H^* = \bigcup \{H(B)\colon B \in \Gamma\}$ is metacompact.

Suppose that $\mathscr{G}$ is any open cover of X. Then $\mathscr{G}|H^*$ has a point finite open (in H^*) refinement $\mathscr{V}$. Since X is almost expandable, by

Theorem 2.12 of [21] there exists a point finite open (in X) collection $\mathscr{W}_1$ such that $\mathscr{W}_1|H^*$ refines $\mathscr{V}$. We may also assume that $\mathscr{W}_1$ refines $\mathscr{G}$ as well. Since $X-H^*$ is metacompact by our induction assumption, $\mathscr{G}|X-H^*$ has a point finite open refinement $\mathscr{W}_2$. Clearly, $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ is a point finite open refinement of $\mathscr{G}$, and hence X is metacompact.

From the previous results we now can easily obtain a result originally due to R. E. Hodel.

COROLLARY 5.2. *The Locally Finite Sum Theorem is true for metacompactness.*

Proof. Suppose

$$X = \bigcup_{\alpha\in A} F_\alpha,$$

where $\mathscr{F} = \{F_\alpha\colon \alpha\in A\}$ is a locally finite collection of closed metacompact subsets of X. Then X is countably metacompact by Theorem 5.3.

Let $U_i = \{x\colon \operatorname{ord}(x,\mathscr{F}) \leqslant i\}$ for each i. Since $\{U_i\}_{i=1}^\infty$ is a monotone open cover of X, there exists a closed cover $\{H_i\}_{i=1}^\infty$ satisfying $H_i \subseteq U_i$ for each i. Now $\mathscr{F}_i = \{F_\alpha \cap H_i\colon \alpha\in A\}$ has the property that each $x\in X$ has a neighborhood which intersects at most i members of $\mathscr{F}_i$. Therefore, by Theorem 5.4,

$$F_i^* = \bigcup_{\alpha\in A} [F_\alpha \cap H_i]$$

is metacompact for each i. By Theorems 5.2 and 5.3, X is θ-refinable and almost expandable. Therefore X is metacompact by Theorem 4.3 of [20].

Analogous results for the other generalized paracompact spaces follow in a similar fashion.

OPEN QUESTIONS. (1) Can the countable paracompactness conditions in Theorems 4.1, 4.2, and 4.3 be removed? (**P 966**)

(2) Does subparacompactness have a characterization analogous to Corollary 3.1? (**P 967**)

(3) Do other properties like collectionwise normality have characterization in terms of sequences of open covers? (**P 968**)

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