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SOME PROBLEMS CONCERNING STABILITY OF FIXED POINTS

BY

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1. Introduction. If (X, d) is a metric space, then a function $f\colon X \to X$ is called a *contraction mapping* (with respect to the metric d) if and only if there is a real number a, $0 \leqslant a < 1$, such that $d\big(f(x), f(y)\big) \leqslant ad(x, y)$ for all $x, y \in X$. The real number a is called a *Lipschitz constant for f*. The following stability result was proved in [3], Theorem 2, and extended in [2] and [4]:

THEOREM 1.1. *Let (X, d) be a locally compact metric space. Then*

(γ) *if $f_i\colon X \to X$ is a contraction mapping with fixed point a_i for each $i = 0, 1, 2, \ldots$ and if the sequence $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f_0, then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_0.*

This result was principally motivated in [3] by my desire to determine (see Section 2 of [3] and Theorem 2.5 in this paper) certain kinds of mappings of cartesian products which have fixed points. It has also been applied to the area of stability of solutions to differential equations (see, for example, [3]) and to various types of problems in functional analysis.

Over the past several years, I have become interested in (γ) as a property "in itself". The purpose of this paper* is to formally consider (γ) as a property of spaces and to pose a number of problems concerning (γ). We will state some theorems and indicate some proofs, but only when they enhance or are related to problems which are posed.

2. We say that a metric space (X, d) has *property* c, written $(X, d) \in \operatorname{prop}(c)$, if and only if (X, d) satisfies (γ). We point out that property c is not, in general, a topological property but depends on the metric. This is because the functions which are contraction mappings on a space may change (even) with a change to a topologically equivalent metric. We also point out that completeness is not an *a priori* requirement in order that we consider whether or not a space has property c. The classical result of Banach guarantees that a contraction mapping defined on a com-

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plete space (into itself) has a (unique) fixed point; however, in determining whether a space has property c, we only "use" those contraction mappings which have fixed points.

At first glance, it might not be clear that there are metric spaces which do not have property c. However, we gave classes of Banach spaces not having property c in [3] and we will give other examples in this paper.

Theorem 1.1 can now be restated as follows:

THEOREM 2.1. *If* (X, d) *is a locally compact metric space, then* $(X, d) \in \text{prop}(c)$.

It is not difficult to construct some specific metric spaces with property c which are not locally compact. The following theorem yields a class of such spaces:

THEOREM 2.2. *Any separable metric space can be remetrized with a topologically equivalent metric so as to have property* c.

The basic idea in proving Theorem 2.2 is to embed the space in the Hilbert cube and take the induced metric; the resulting space is totally bounded. Since pointwise convergence of a sequence of contraction mappings in a totally bounded metric space is equivalent to uniform convergence, it follows from Theorem 1 of [3] that a totally bounded metric space has property c.

PROBLEM 2.1. Can any metric space be remetrized with a topologically equivalent metric so as to have property c? (**P 833**)

Theorem 2.1 can obviously be interpreted as saying that property c is a topological invariant for the class of locally compact metric spaces. Since there are lots of separable metric spaces which do not have property c, Theorem 2.2 seems to suggest that the class of metric spaces for which property c is a topological property cannot be "too large". We have the following

PROBLEM 2.2. For what metric spaces is property c a topological invariant? Is it true, in fact, that if a metric space has property c with every topologically equivalent metric, then the metric space must be locally compact? (**P 834**)

Theorem 2.2 shows that local compactness is, for the class of all metric spaces, an extremely strong sufficient condition for property c. However, for the class of Banach spaces, the relation between local compactness (i.e., finite dimensionality) and property c seems quite different. In [3], Theorem 3, the following result was proved:

THEOREM 2.3. *A separable or reflexive Banach space is finite dimensional if and only if it has property* c.

This result leads to the following problem, which we consider to be the most interesting problem mentioned here:

PROBLEM 2.3. Does property c characterize finite dimensionality for the class of all Banach spaces? Of course, in view of Theorem 2.1, this question reduces to: If a Banach space has property c, must it be finite dimensional? (**P 835**)

The techniques in [3] used in proving Theorem 2.3 stated above lead to the next two questions which are auxiliary to Problem 2.3.

PROBLEM 2.4. Is it enough to consider affine maps to determine if a Banach space has property c? (**P 836**) By an affine map we mean a translation, by a vector, of a linear map of the Banach space.

PROBLEM 2.5. Determine whether or not, for every infinite dimensional Banach space, there is a sequence of linear functionals of norm one which is weak * convergent to the zero linear functional (**P 837**).

If there are always such linear functionals, then the methods in [3] can be used to solve Problems 2.3 and 2.4 affirmatively. We wish to mention that Professor R. B. Fraser, Jr., and Professor Dorothy Stone have communicated to the author some examples of non-separable, non-reflexive Banach spaces for which there is such a sequence of linear functionals.

Problem 2.3 can clearly be restated for the more general class of metric linear spaces. However, we do not even know the answer when the metric linear space is separable. In particular, let (s, ϱ) denote the countable product of real lines, where

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

for each $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in s$.

PROBLEM 2.6. Does (s, ϱ) have property c? (**P 838**)

Since s is a product space, Problem 2.6 suggests a more general problem. Take as the metric for a finite or countably infinite cartesian product of metric spaces any one of the standard product metrics. Then we have

PROBLEM 2.7. Does the cartesian product of two or of countably infinitely many metric spaces with property c have property c? Does the countably infinite product of locally compact metric spaces have property c? (**P 839**)

We do not even know if the cartesian product of a metric space with property c and a locally compact metric space has property c. We do have the following result:

THEOREM 2.4. *The cartesian product of a complete metric space* (X, d_1) *with property* c *and a compact metric space* (Y, d_2) *has property* c, *provided the metric* d *chosen for the cartesian product satisfies*

(a) $d_1((x, z)) \leqslant d((x, y), (z, w))$ *for all* $(x, y), (z, w) \in X \times Y$,

(b) $d((x, y), (z, y)) \leqslant d_1(x, z)$ *for all* $(x, y), (z, y) \in X \times Y$.

Now we turn to property c as it relates to subspaces. Using Lipschitz extensions of contraction mappings, it is not difficult to prove that a dense subspace (with the inherited metric) of a complete metric space with property c has property c; this result has been proved, independently of the author, by Professor Norman Rehner. Again, using Lipschitz extensions, together with appropriate modifications of the proof of Theorem 2 in [3], it follows that any subspace of a locally compact metric space has property c (the fact that a totally bounded metric space has property c can be viewed as a special case of this; cf. the lines immediately following Theorem 2.2).

Several questions on the relation of spaces, subspaces and property c arise.

(1) If a subspace of a (complete) metric space has property c, does its closure? In particular, does the completion of a metric space with property c have property c?

(2) If A and B are two subspaces of a metric space and if $A \in \text{prop}(c)$ and $B \in \text{prop}(c)$, then must $(A \cup B) \in \text{prop}(c)$?

(3) If a metric space has property c, then does every subspace?

We now give an example which shows that the answer to questions (1), (2) and (3) is no.

Example 2.1. Let H be the Hilbert space of all square summable sequences of real numbers with the usual distance d. Let θ be the zero vector in H (i.e., $\theta = (0, 0, \ldots, 0, \ldots)$) and, for each $n = 1, 2, \ldots$, let $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the 1 appears in the n-th coordinate, and let $J_n = \{t \cdot e_n\colon 0 \leqslant t \leqslant 1\}$. We write $x < y$ for x and y in a given J_n provided the n-th coordinate of x is less than the n-th coordinate of y. Now, for each $n = 1, 2, \ldots$, let $X_n \subset J_n$, $X_n = \{\theta = x_1^n < x_2^n < \ldots < x_{k(n)}^n = e_n\}$, be a finite number of points "contracting out" towards e_n (i.e., $d(x_{j+1}^n, x_{j+2}^n) < d(x_j^n, x_{j+1}^n)$ for all $j = 1, 2, \ldots, k(n)-2$) such that $d(\theta, x_2^n) < 1/n$. Let

$$X = \bigcup_{n=1}^{\infty} X_n$$

with the metric for X obtained by restricting d; we denote the restricted metric for X again by d. The metric space (X, d) does not have property c. To see this, first let $\alpha_i\colon X \to X_i$ and $\beta_i\colon X_i \to X_i$ for each $i = 1, 2, \ldots$ be given by

$$\alpha_i(x) = \begin{cases} \theta & \text{if } x \notin X_i, \\ x & \text{if } x \in X_i, \end{cases} \quad \text{and} \quad \beta_i(x_j^i) = \begin{cases} x_{j+1}^i, & j < k(i), \\ x_j^i, & j = k(i). \end{cases}$$

It is easy to verify that, for each $i = 1, 2, \ldots, d(\alpha_i(x), \alpha_i(y)) \leqslant d(x, y)$ for all $x, y \in X$, and β_i is a contraction mapping. Hence, letting $f_i = \beta_i \circ \alpha_i$ for each $i = 1, 2, \ldots$, we see that each f_i is a contraction mapping with fixed point $x^i_{k(i)} = e_i$. Let f_0: $X \to X$ be given by $f_0(x) = \theta$ for all $x \in X$. Since $\{x^i_2\}_{i=1}^{\infty}$ converges to θ, it is not difficult to see that the sequence $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f_0. From this we infer that (X, d) does not have property c. Since $X - \{\theta\}$ is locally compact, $(X - \{\theta\}) \in \text{prop(c)}$. The negative answer to questions (1) and (2) above-mentioned follows now easily. To see that question (3) has a negative answer, we add some points to X to obtain a new space Y with property c. We will only describe how to form the space Y, leaving all other details to the reader. For each $n = 1, 2, \ldots$, let Y_n be a finite number of points in J_n such that $Y_n \supset X_n$ and the points in Y_n "contract in" towards θ. Let

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

(with the metric for H restricted). Note that, since X is a closed subspace of Y, property c is not hereditary for closed subspaces.

PROBLEM 2.8. What conditions, besides local compactness, imply that every (closed) subspace of a metric space with property c has property c? If every subspace of a metric space (X, d) has property c, then what must be true about the space (X, d)? (**P 840**) We point out that (X, d) need not be locally compact.

As mentioned in the introduction, the principal motivation for proving Theorem 2 of [3] was to determine certain kinds of mappings of cartesian products which have fixed points. Recall that a space is said to have the *fixed point property* if and only if every continuous self-map has a fixed point. The same proof as that given for part (2) of Theorem 4 in [3] proves the following more generally stated result:

THEOREM 2.5. *Let (X, d_1) be a complete metric space, let (Y, d_2) be a metric space with the fixed point property, and let f be a continuous function from $X \times Y$ into $X \times Y$, where the metric d for $X \times Y$ satisfies* (a) *and* (b) *of Theorem* 2.4. *If (X, d_1) has property* c *and if f is a contraction mapping in the first variable, then f has a fixed point.*

We remark that this result, in general, would be false without the requirement that (Y, d_2) have the fixed point property (see [3]). However, we do not know the answer to the following problem which was posed to the author by Professor R. B. Fraser, Jr. [1]:

PROBLEM 2.9. Let (X, d_1) and (Y, d_2) be compact metric spaces and let d be a metric for $X \times Y$ which satisfies (a) and (b), with respect to

[1] Some progress on the problem has recently been made by Professor Haskell Cohen (*added in proof*).

both d_1 and d_2, of Theorem 2.4. If $f\colon X\times Y\to X\times Y$ is a contraction mapping in each variable separately, then does f have a fixed point?

Of course, if either of the coordinate spaces has the fixed point property, then the question has an affirmative answer by applying Theorem 2.5. R. B. Fraser, Jr., has pointed out that if the metric d for the product space is taken to be the "taxi cab" metric (i.e., $d\big((x,y),(z,w)\big) = d_1(x,z)+d_2(y,w)$), then the question has an affirmative answer because the conditions on f imply, in this case, that f is contractive [1] and a result due to Edelstein (see 3.1 of [1]) can be applied.

Theorem 1.1 of this paper remains valid if the word "contraction" is replaced by the word "contractive" (see Theorem 1 of [2]). Let us say that a metric space has *property* c′ if and only if it satisfies (γ) for the more general contractive mappings. The problems mentioned in this paper are also unsolved with property c′ replacing property c. In addition, we have the following problem:

PROBLEM 2.10. Is there a metric space with property c which does not have property c′? If so, then for what classes of metric spaces are these two properties equivalent? (**P 841**)

We conclude this paper with a problem not involving property c. In [5], Smart observed that if a compact metric space has the property that the identity mapping is a uniform (or, what is equivalent, pointwise) limit of contraction mappings, then every non-expansive mapping has a fixed point.

PROBLEM 2.11. If a compact metric space (X, d) has the property that

(*) the identity mapping is a pointwise limit of contraction mappings,

then must $\check{H}^n(X,Z) = 0$ for all n (where $\check{H}^n(X,Z)$ denotes the n-th Čech cohomology group of X over the integers Z)? (**P 842**)

We point out that a compact metric space satisfying (*) must be connected.

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