COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 2

PRINCIPAL PROJECTION BANDS OF A RIESZ SPACE

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In this note we give a solution to a problem on bands of a Riesz space proposed by Luxemburg and Zaanen [5], and we investigate some analogous problems for lattice ordered groups.

For the fundamental concepts of vector lattices and lattice ordered groups see Birkhoff [1], Conrad [2], and Fuchs [3]. Note that in [5] the Bourbaki terminology is used, and a vector lattice is called a *Riesz space* (in [4] and [7] a vector lattice is called a *K-lineal*).

1. Riesz spaces. Let us recall the following notions. Let L be a Riesz space and A a linear subspace of L such that

(a) if $X \subset A$ and X has a supremum in L, then this supremum belongs to A;

(b) if $a \in A$, $b \in L$, $|b| \leqslant |a|$, then $b \in A$.

Under these assumptions A is said to be a *band* of L. From (a) and from the fact that A is a linear subspace of L it follows that if $Y \subset A$ and $\inf Y$ exists in L, then $\inf Y \in A$. Let $c \in L$. The smallest band of L containing the element c is called a *principal band generated by the element c.*

Let $X \subset L$. Write

$$X^{\delta} = \{y \in L: |x| \wedge |y| = 0 \text{ for each } x \in X\}.$$

Then X^{δ} is a band of L ([5], Theorem 19.2).

Let A and B be bands of L such that the Riesz space L is an order direct sum of A and B (cf. [5], § 24); we write

$$L = A \oplus B. \tag{1}$$

Then A and B are called *projection bands* in L. A band A in L is a projection band in L if and only if for each u, $0 < u \in L$, the element

$$u_1 = \sup\{v \in A: 0 < v \leqslant u\} \tag{2}$$

exists in L (cf. [5], Theorem 24.5). If (1) holds and f is an element of L, then the components of f in A (respectively, in B) will be denoted by $f(A)$ (respectively, by $f(B)$). It is easy to verify that by (1) we have $B = A^\delta$ (cf. [5], Theorem 24.1).

A projection band of L that is at the same time a principal band will be called a *principal projection band.* A constructive characterization of principal projection bands is given in [5], Theorem 24.7.

The following two notions were introduced and studied in [5].

Definition 1. A Riesz space L is said to have *sufficiently many projections* if every non-zero band of L contains a non-zero projection band of L.

Definition 2. A Riesz space L is said to have *property* (o.d.) if every non-zero projection band of L contains a non-zero principal projection band of L.

In other words, property (o.d.) means that the set of all principal projection bands is order dense in the set of all projection bands (partially ordered by the set-theoretical inclusion).

In [5], p. 183, the question has been proposed whether the property to have sufficiently many projections implies property (o.d.). We shall show that the answer to this question is positive.

Remark. It follows from Theorems 22.3 and 30.4 of [5] that if a Riesz space L has sufficiently many projections, then for each band A of L there exists $X \subset L$ with $(X^\delta)^\delta = A$. (This result will not be used in what follows.)

LEMMA 1. *Let C be a band in L and let formula* (1) *be valid. Then* $C = (A \cap C) \oplus (B \cap C)$.

Proof. It suffices to verify that for each $c \in C$ we have $c(A) \in A \cap C$ and $c(B) \in B \cap C$. Put $c \vee 0 = u$, and $-(c \wedge 0) = z$. Then $c = u - z$. Let u_1 be as in (2). Clearly, $u \in C$ and $0 \leqslant u_1 \leqslant u$. Thus by (b) we obtain $u_1 \in C$. According to Theorem 24.5 of [5], $u_1 = u(A)$. Hence $u(A) \in A \cap C$. Analogously, we get $z(A) \in A \cap C$, and hence $c(A) = u(A) - z(A) \in A \cap C$. Similarly, $c(B) \in B \cap C$.

LEMMA 2. *Let* (1) *be valid. Let A_1 and B_1 be bands in A and B, respectively. Then $L_1 = A_1 \oplus B_1$ is a band in L.*

Proof. L_1 is a linear subspace of the space L. Let $X \subset L_1$, and $\sup X = x_0 \in L$. Write

$$X_1 = \{x(A) \colon x \in X\} \quad \text{and} \quad X_2 = \{x(B) \colon x \in X\}.$$

Then by (1) we obtain $x_0(A) = \sup X_1$, and $x_0(B) = \sup X_2$. It follows from (1) that A is a band in L, and hence A_1 is a band in L. Clearly, $X_1 \subset A_1$, thus $\sup X_1 \in A_1$. Hence $x_0(A) \in A_1$ and, analogously, $x_0(B) \in B_1$. Therefore,

$$\sup X = x_0 = x_0(A) + x_0(B) \in L_1.$$

Hence (a) is valid for the set L_1. Let $c \in L_1$, $d \in L$, $|d| \leqslant |c|$. Clearly, $|x|(A) = |x(A)|$ for each $x \in L$. Thus by $|d| \leqslant |c|$ we obtain $|d(A)| \leqslant |c(A)|$. Since $c \in L_1$, we have $|c(A)| \in A_1$, and since A_1 is a band in A, we infer that $d(A) \in A_1$. Analogously, we verify that $d(B) \in B_1$. Thus $d = d(A) + d(B) \in L_1$. Therefore, L_1 is a band in L.

LEMMA 3. *Let C be a principal band in L generated by an element c, and let A be a projection band in L such that $A \subset C$. Then A is a principal band in L.*

Proof. Since A is a projection band, there is a band B in L such that (1) is valid. Hence, according to Lemma 1,

$$C = A \oplus (B \cap C). \tag{3}$$

We intend to show that A is a principal band in L generated by the element $c(A)$.

If A is not a principal band in L generated by $c(A)$, then, because of $c(A) \in A$, there exists a band A_1 in L such that A_1 is a proper subset of A and $c(A) \in A_1$. Put $C_1 = A_1 \oplus (B \cap C)$. By Lemma 2, C_1 is a band in L. We have already shown (cf. the proof of Lemma 1) that $c(B) \in B \cap C$. Hence

$$c = c(A) + c(B) \in A_1 \oplus (B \cap C).$$

Since A_1 is a proper subset of A, it follows from (3) that C_1 is a proper subset of C. Thus C is not the least band containing the element c, which is a contradiction.

THEOREM 1. *If L is a Riesz space having sufficiently many projections, then it has property* (o.d.).

Proof. Let P be a non-zero projection band in L. Then there is an element p, $0 < p \in P$. Let Q be the principal band in L generated by p. We have $\{0\} \neq Q \subset P$. Since L has sufficiently many projections, there is a projection band A in L such that $\{0\} \neq A \subset Q$. According to Lemma 3, A is a principal band. Hence $A \subset P$ and A is a non-zero principal projection band. This shows that L has property (o.d.).

2. Lattice ordered groups. Let G be a lattice ordered group. The group operation will be denoted by $+$, though we do not assume the commutativity of this operation. Let $X \subset G$, and $y \in G$. Analogously as in Section 1, we write

$$X^{\delta} = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\}. \tag{4}$$

The set X^{δ} will be called a *polar* of G. We put $(X^{\delta})^{\delta} = X^{\delta\delta}$. The set $\{y\}^{\delta\delta}$ is said to be a *principal polar* of G. The polars of a lattice ordered group have been investigated in [6]. Each polar X^δ of G is an l-subgroup of G; moreover, X^δ is a convex sublattice of G, i.e., if $z_1, z_2 \in X^\delta$, $g \in G$, and $z_1 \leqslant g \leqslant z_2$, then $g \in X^\delta$ (cf. [6]).

Let A and B be l-subgroups of G. Assume that G is a direct sum of A and B in the group-theoretical sense (with respect to the additive notation we use the term "direct sum" rather than the term "direct product") and that

$$0 \leqslant g \in G, \; a \in A, \; b \in B, \; g = a+b \text{ imply } a \geqslant 0, \; b \geqslant 0.$$

Then we write

$$G = A \oplus B; \tag{5}$$

A and B are called *direct summands* of the lattice ordered group G. The component of an element $g \in G$ in A (respectively, in B) will be denoted by $g(A)$ (respectively, by $g(B)$). If (5) is valid, then $B = A^\delta$ (cf., e.g., [2], 2.3, Proposition 8). From the definition of a direct sum it follows immediately that if A is a direct summand of G, then

(i) $g_1, g_2 \in G$, $g_1 \leqslant g_2 \Rightarrow g_1(A) \leqslant g_2(A)$,

(ii) $g_1 \in A \Leftrightarrow g_1(A) = g_1 \Leftrightarrow g_1(A^\delta) = 0$,

(iii) $0 \leqslant g_1 \in G \Rightarrow 0 \leqslant g_1(A) \leqslant g_1$.

By (ii) and (iii), for $0 \leqslant g \in G$ we obtain

$$g(A) = \sup\{a \in A \colon 0 \leqslant a \leqslant g\} \tag{6}$$

(for the analogous result concerning Riesz spaces cf. (2) and [5], Theorem 24.5).

An l-subgroup C of G is said to be *closed* in G if, whenever $C_1 \subset C$ and $\sup C_1$ exists in G, then $\sup C_1 \in C$. Let $g \in G$ and let $C(g)$ be the set of all closed convex l-subgroups of G containing the element g. Then the l-subgroup $C_0 = \bigcap G_i$ $(G_i \in C(g))$ will be called a *principal closed convex l-subgroup* of G generated by g.

A principal closed convex l-subgroup of G that is at the same time a direct summand of G is said to be a *principal direct summand* of G.

Let us consider the following conditions for G:

(α) Each non-zero closed convex l-subgroup of G contains a non-zero direct summand of G.

(β) Each non-zero polar of G contains a non-zero direct summand of G.

(γ) Each non-zero direct summand of G contains a non-zero principal direct summand of G.

(δ) If A is a non-zero direct summand of G, then A contains a non-zero direct summand A_0 of G such that A_0 is a principal polar of G.

We shall investigate relations between those conditions; the result can be expressed by the diagram

$$\begin{tikzcd} & (\beta) \arrow[dr, Rightarrow] & \\ (\alpha) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (\delta) \\ & (\gamma) \arrow[ur, Rightarrow] & \end{tikzcd}$$

where $\Rightarrow$ denotes the implication.

We have already remarked that each polar of G is convex in G; moreover, each polar of G is closed in G (cf. [6]). Hence $(\alpha) \Rightarrow (\beta)$.

LEMMA 4. *Let A be a principal polar of G and let B be a direct summand of G such that $B \subset A$. Then B is a principal polar of G.*

Proof. According to the assumption there is $a \in G$ with $A = \{a\}^{\delta\delta}$. Without loss of generality we may suppose that $a \geqslant 0$. From the definition of the polar (cf. (4)) it follows immediately that for any X and $Y \subset G$ we have

$$X \subset Y \Rightarrow X^{\delta} \supset Y^{\delta}. \tag{5'}$$

Let $P(G)$ be the set of all polars of G and let $P(G)$ be partially ordered by the inclusion. Šik [6] has proved that $P(G)$ is a Boolean algebra and that, for each $Z \in P(G)$, the set Z^{δ} is a complement of Z in $P(G)$. Therefore,

$$X^{\delta\delta\delta} = X^{\delta} \tag{6'}$$

holds for each $X \subset G$.

Since B is a direct summand of G, it is a polar of G, and hence there is $X \subset G$ with $X^{\delta} = B$. Thus by (6') we obtain $B^{\delta\delta} = B$.

Put $a(B) = b$. Then $\{b\} \subset B$, and hence, by using (5') twice, we get

$$\{b\}^{\delta\delta} \subset B^{\delta\delta}. \tag{7}$$

We intend to show that the relation

$$B^{\delta\delta} \supset \{b\}^{\delta\delta} \tag{8}$$

is valid. By (5') and (6'), relation (8) implies

$$B^{\delta} \supset \{b\}^{\delta} \tag{9}$$

and, conversely, from (5') it follows that (9) implies (8). Hence it suffices to verify that (9) is valid.

For any subset Z of G put $Z^{+} = \{z \in Z : z \geqslant 0\}$. Let Z_1 and Z_2 be l-subgroups of G with $Z_1^{+} \subset Z_2^{+}$. Let $z \in Z_1$ and write $z_1 = z \vee 0$, $z_2 = -(z \wedge 0)$. Then $z_1, z_2 \in Z_1^{+}$ and $z = z_1 - z_2 \in Z_2$. Hence $Z_1 \subset Z_2$. Therefore, if T_1 and T_2 are l-subgroups of G such that $T_1 \not\subset T_2$, then $T_1^{+} \not\subset T_2^{+}$.

Suppose that relation (9) does not hold. Then, since B^{δ} and $\{b\}^{\delta}$ are l-subgroups of G, there is an element g, $0 < g \in \{b\}^{\delta}$, such that $g \notin B^{\delta}$.

Since B is a direct summand of G, we have

$$G = B \oplus B^\delta, \quad g = g(B) + g(B^\delta).$$

By (iii), $g \geqslant g(B) \geqslant 0$, and $g \geqslant g(B^\delta) \geqslant 0$. Since $g \notin B^\delta$, we infer from (ii) that $g(B) > 0$. Write $g(B) = g_1$. Since $\{b\}^\delta$ is a convex l-subgroup of G and $g \in \{b\}^\delta$, we get $g_1 \in \{b\}^\delta$.

Clearly, $g_1 \in B \subset A$. Since $A = \{a\}^{\delta\delta}$, $a > 0$, we must have $g_1 \wedge a > 0$. Let us put

$$g_1 \wedge a = g_2. \tag{10}$$

Then $0 < g_2 \leqslant g_1$ and, since B is a convex sublattice of G, we obtain $g_2 \in B$. Hence, by (ii), $g_1(B) = g_1$ and $g_2(B) = g_2$. By (10) we get

$$g_1(B) \wedge a(B) = g_2(B),$$

whence

$$g_1 \wedge b = g_2 > 0.$$

Therefore, $g_1 \notin \{b\}^\delta$, which is a contradiction. Thus relation (9) is valid. This implies that $B = B^{\delta\delta} = \{b\}^{\delta\delta}$.

Remark. The assertion of Lemma 4 fails to hold if we suppose only that B is a polar of G. For example, let G_0 be the set of all real functions defined on the interval $[0, 1]$. For $f, g \in G_0$ we put $f \leqslant g$ if $f(x) \leqslant g(x)$ for each $x \in [0, 1]$. Then G_0 is an additive lattice ordered group. Let K be the set of all constants of G_0 (i.e., the set of all $f \in G_0$ such that $f(t_1) = f(t_2)$ for each pair $t_1, t_2 \in [0, 1]$). We denote by f^1 the function such that $f^1 \in K$ and $f^1(0) = 1$. For every $x \in [0, 1]$ let $f_x \in G_0$ be such that $f_x(x) = 1$ and $f_x(t) = 0$ for each $t \in [0, 1]$, $t \neq x$. Let G be the l-subgroup of G_0 generated by the set $K \cup \{f_x : x \in [0, 1]\}$. Put $A = G$. Then $A = \{f^1\}^{\delta\delta}$, whence A is a principal polar of G. Let $M = \{f_x : x \in [0, \frac{1}{2}]\}$, and $B = M^\delta$. We have $B \subset A$ and the polar B is not principal in G.

The proof of the following theorem is analogous to that of Theorem 1.

THEOREM 2. *For each lattice ordered group G, condition* (β) *implies* (δ).

Proof. Let G be a lattice ordered group fulfilling (β) and let A be a non-zero direct summand of G. There is an a, $0 < a \in A$. Therefore, by (5′), $\{0\} \neq \{a\}^{\delta\delta} \subset A^{\delta\delta}$. Since A is a direct summand, it is a polar, and thus $A^{\delta\delta} = A$. Since G fulfils (β), there is a direct summand B of G such that $\{0\} \neq B \subset \{a\}^{\delta\delta}$. By Lemma 4, there is $b \in B$ such that $B = \{b\}^{\delta\delta}$. Hence G satisfies condition (δ).

COROLLARY 1. *For each lattice ordered group G, condition* (α) *implies* (δ).

THEOREM 3. *Let G be a lattice ordered group fulfilling* (α), *and let A be a non-zero direct summand of G. Then the following conditions are equivalent:*

(a) *A is a principal polar in G;*

(b) *A is a principal closed convex l-subgroup in G.*

Proof. Suppose that A is a principal closed convex l-subgroup of G generated by an element a. Since A is a direct summand, it is a polar of G and hence $A^{\delta\delta} = A$. From $a \in A$ and from (5′) we infer that $\{a\}^{\delta\delta} \subset A$. Since $\{a\}^{\delta\delta}$ is a closed convex l-subgroup of G containing a, we have $A \subset \{a\}^{\delta\delta}$. Thus $A = \{a\}^{\delta\delta}$ is a principal polar in G.

Assume that A is a principal polar in G, $A = \{a\}^{\delta\delta}$. Let A_1 be the principal closed convex l-subgroup of G generated by the element a. Then $A_1 \subset A$. Since G fulfils (α), there exists a non-zero direct summand B contained in A_1. Let S be a system of non-zero direct summands of G contained in A_1. The system S will be said to be *maximal disjoint* provided that

(i) $B_1 \cap B_2 = \{0\}$ for each pair of distinct elements B_1, B_2 of S;

(ii) if B_0 is a non-zero direct summand of G contained in A_1 and if $B_0 \notin S$, then there is $B_1 \in S$ such that $B_1 \cap B_0 \neq \{0\}$.

From the Zorn Lemma it follows that there exists a maximal disjoint system $S = \{B_i: i \in I\}$ of non-zero direct summands of G contained in A_1.

Without loss of generality we may suppose that $a > 0$. Put $a_i = a(B_i)$ for $i \in I$. Then $0 \leqslant a_i \leqslant a$ for each $i \in I$. Suppose that there exists $a' \in G$ such that $a_i \leqslant a' < a$ for each $i \in I$. From the convexity of A_1 it follows that $a' \in A_1$, and hence $0 < a - a' = a'' \in A_1$. For each $i \in I$ we have

$$a_i = a_i(B_i) \leqslant a'(B_i) \leqslant a(B_i) = a_i,$$

whence $a'(B_i) = a(B_i)$ and thus $a''(B_i) = 0$. By this and by (6) we obtain $|b_i| \wedge a'' = 0$ for each $b_i \in B_i$ and each $i \in I$; therefore

$$B_i \subset \{a''\}^{\delta\delta} \quad \text{for each } i \in I. \tag{11}$$

Write $C = \{a''\}^{\delta\delta} \cap A_1$. The set C is a closed convex l-subgroup of G and $a'' \in C$, whence $C \neq \{0\}$. Thus, according to (α), there is a non-zero direct summand B_0 of G contained in C. From (11) it follows that $B_i \cap C = \{0\}$ for each $i \in I$ and, consequently, $B_i \cap B_0 = \{0\}$ for each $i \in I$, and $B_0 \subset A_1$. This contradicts the maximality of S. Hence no element $a' \in G$ with the above-mentioned properties can exist and, therefore,

$$a = \bigvee a_i \quad (i \in I). \tag{12}$$

Let $0 < x \in A$. Put $x_i = x(B_i)$ for each $i \in I$. We have $0 \leqslant x_i \leqslant x$. Suppose that there is $x' \in G$ such that $x_i \leqslant x'$ for each $i \in I$, and $x' < x$. Put $x - x' = x''$. Analogously as for the element a'', we can now verify that

$$B_i \subset \{x''\}^{\delta} \tag{11′}$$

is valid for each $i \in I$. From (11′) it follows that $a_i \wedge x'' = 0$ holds for each $i \in I$, and hence, by (12),

$$a \wedge x'' = \left(\bigvee a_i\right) \wedge x'' = \bigvee (a_i \wedge x'') = 0.$$

Hence $x'' \in \{a\}^{\delta}$ and thus, since $x'' \neq 0$, $x'' \notin \{a\}^{\delta\delta} = A$, which is a contradiction. Therefore

$$\bigvee x_i = x. \tag{13}$$

Since $x_i \in B_i \subset A_1$ and A_1 is a closed l-subgroup of G, we infer from (13) that $x \in A_1$. Hence $A = A_1$, and so A is a principal closed convex l-subgroup of G.

By Theorem 3 and by Corollary 1 we obtain

COROLLARY 2. *For each lattice ordered group G, condition* (α) *implies* (γ).

Remarks. 1. Condition (β) does not imply (α). For example, let $G = R \circ R$, where the symbol $\circ$ denotes the lexicographic product (cf. [3]) and R is the additive group of all reals with the natural linear order. Since G is linearly ordered, each non-zero polar in G coincides with G, and so does each non-zero direct summand in G; thus (β) is valid. There exists a closed convex l-subgroup of G that is distinct from G; hence (α) fails to hold.

2. Condition (β) does not imply (γ). For example, let Λ be the set of all negative integers (with the natural order). For each $\lambda \in \Lambda$ let $A_\lambda = R$, and let $G = \Gamma_{\lambda \in \Lambda} A_\lambda$ be the lexicographic product of linearly ordered groups A_λ (cf. [3]). Again, G is linearly ordered, and hence each non-zero polar of G coincides with G; thus (β) holds. Let $0 \neq a \in G$. There is $\lambda_0 \in \Lambda$ such that $a(\lambda_0) \neq 0$ and $a(\lambda) = 0$ for each $\lambda \in \Lambda$, $\lambda < \lambda_0$. Then the principal closed convex l-subgroup of G generated by the element a consists of all elements $b \in G$ with $b(\lambda) = 0$ for each $\lambda < \lambda_0$. Thus each principal closed convex l-subgroup of G is distinct from G; therefore, (γ) does not hold.

3. The same example shows that (δ) does not imply (γ).

4. Condition (γ) implies (δ). In fact, suppose that (γ) holds for G and let A be a non-zero direct summand of G. According to (γ), there exists a direct summand B of G with $B \subset A$ such that B is a principal closed convex l-subgroup of G generated by an element b, $0 \neq b \in B$. Since $\{b\}^{\delta\delta}$ is a closed convex l-subgroup of G containing b, we have $B \subset \{b\}^{\delta\delta}$. On the other hand, because B is a polar and $b \in B$, we get $\{b\}^{\delta\delta} \subset B$. Thus $B = \{b\}^{\delta\delta}$ and (δ) is valid.

5. Let L be a Riesz space. Then L can be considered as a lattice ordered group (if we disregard the multiplication of elements of L by reals); hence all notions introduced for lattice ordered groups can be applied for L. If A is a band in L, then, clearly, A is a closed convex l-subgroup of L. Let A_1 be a closed convex l-subgroup of L considered as lattice ordered group. To verify that A_1 is a band in the Riesz space L it suffices to show that if $a \in A_1$ and $\alpha \in R$, then $\alpha a \in A_1$. Let n be a positive integer with $n > |\alpha|$. Then $n|a| \in A_1$ and $-n|a| \leqslant \alpha a \leqslant n|a|$. Hence, by

the convexity of A_1, the element aa belongs to A_1 and, therefore, A_1 is a band in L. Obviously, each projection band in L is a direct summand in L considered as a lattice ordered group. Let B be a direct summand of the lattice ordered group L. Then $B^{\delta\delta} = B$, and thus B is a closed convex l-subgroup of L and B is a band of L. Moreover (cf. (6) and (2)), B is a projection band in L. Thus Corollary 2 yields a new proof of Theorem 1.

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Reçu par la Rédaction le 28. 9. 1973;
en version modifiée le 5. 12. 1974