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CHARACTERIZATIONS OF THE SPHERE BY THE CURVATURE OF THE SECOND FUNDAMENTAL FORM

BY

THOMAS HASANIS (IOANNINA)

1. On an ovaloid S with Gaussian curvature $K > 0$ in a Euclidean 3-dimensional space E^3, the second fundamental form II defines a positive-definite Riemannian metric, if appropriately oriented. We denote by K_{II} the Gaussian curvature of the second fundamental form, and by H the mean curvature of S. Many authors were concerned with the problem of characterization of the sphere by the curvature of the second fundamental form. A good result in this direction has been proved in [3]. In [3] it is shown that an ovaloid S in E^3 such that $K_{\mathrm{II}} = cH^sK^r$, where c, s, r are constants and $0 \leqslant s \leqslant 1$, is a sphere. I think that an ovaloid with $K_{\mathrm{II}} = cH^sK^r$, where c, s, and r are constants, is a sphere. The purpose of this paper is to prove the above conjecture in the case where $0 \leqslant r \leqslant \frac{1}{2}$ and c, s are arbitrary constants.

Let Γ_{ij}^k, ∇ and ${}_{\mathrm{II}}\Gamma_{ij}^k$, ∇_{II} denote the Christoffel symbols and the first Beltrami operator with respect to the first fundamental form I and to the second fundamental form II, respectively. Then the functions $T_{ij}^k = \Gamma_{ij}^k - {}_{\mathrm{II}}\Gamma_{ij}^k$ are components of a tensor ([1], p. 33). Using the second fundamental tensor b_{ij} for "raising and lowering the indices" we infer ([5], p. 232) that $T_{ijk} = T_{ij}^h b_{hk}$ is totally symmetric and

$$K_{\mathrm{II}} = H + \frac{1}{2} T_{ijk} T^{ijk} - \frac{1}{8K^2} \nabla_{\mathrm{II}} K, \tag{1}$$

which implies easily ([2], p. 7) the equality

$$2H(K_{\mathrm{II}} - H)(H^2 - K) = \frac{1}{2} K \nabla_{\mathrm{II}}\left(H, \frac{H^2}{K}\right) - \frac{1}{4} \nabla\left(\frac{H^2}{K}, K\right). \tag{2}$$

2. First we prove a lemma which generalizes a theorem in [6], p. 240.

LEMMA 2.1. *Let S be a convex surface in E^3 (not necessarily closed). If the function H^2/K attains a relative maximum on S, then there exists a point $P_0 \in S$ such that $K_{\mathrm{II}}(P_0) = H(P_0)$.*

Proof. Let $P_0 \in S$ be a point at which the function H^2/K attains a relative maximum. Then there exists a neighbourhood U of P_0 such that

$$\frac{H^2}{K}(P_0) \geqslant \frac{H^2}{K}(P) \quad \text{for all } P \in U.$$

If $(H^2/K)(P_0) = 1$, then all points of U are umbilics and the neighbourhood U is a piece of sphere, and thus $K_{\mathrm{II}} = H$ on U. Let $(H^2/K)(P_0) > 1$. Since P_0 is a critical point of H^2/K, it follows from (2) that

$$\big(K_{\mathrm{II}}(P_0) - H(P_0)\big)\big(H^2(P_0) - K(P_0)\big) = 0 \quad \text{or} \quad K_{\mathrm{II}}(P_0) = H(P_0)$$

(since $H^2(P_0) > K(P_0)$ by assumption).

The following lemma is well known (see [3] and [6], p. 241). Here we give a new proof of it.

LEMMA 2.2. *Let S be a convex surface (not necessarily closed). If P_0 is a critical point of K or of H, then $K_{\mathrm{II}}(P_0) \geqslant H(P_0)$.*

Proof. If P_0 is a critical point of K, then from (1) we get $K_{\mathrm{II}}(P_0) \geqslant H(P_0)$, since $\frac{1}{2}T_{ijk}T^{ijk} \geqslant 0$. Also, from (2) we obtain easily

(3)
$$(K_{\mathrm{II}} - H)(H^2 - K) = \frac{1}{2}\nabla_{\mathrm{II}}H - \frac{H}{4K}\nabla_{\mathrm{II}}(H, K) - \frac{1}{4K}\nabla(H, K) + \frac{H}{8K^2}\nabla K.$$

If P_0 is a critical point of H, then from (3) we get

$$\big(K_{\mathrm{II}}(P_0) - H(P_0)\big)\big(H^2(P_0) - K(P_0)\big) = \frac{H}{8K^2}(P_0)\nabla K(P_0) \geqslant 0,$$

and thus $K_{\mathrm{II}}(P_0) \geqslant H(P_0)$ because if $K_{\mathrm{II}}(P_0) < H(P_0)$, then $H^2(P_0) = K(P_0)$ or $\nabla K(P_0) = 0$, i.e. P_0 is a critical point of K, and so $K_{\mathrm{II}}(P_0) \geqslant H(P_0)$, a contradiction.

THEOREM 2.1. *Let S be an ovaloid in E^3. If $K_{\mathrm{II}} = cH^sK^r$, where c, s, r are constants and $0 \leqslant r \leqslant \frac{1}{2}$, then S is a sphere.*

Remark. Obviously, by the Gauss-Bonnet theorem, c must be a positive constant. Let dA and dA_{II} denote the area elements of S with respect to the first and second fundamental forms. Then it is obvious that $dA_{\mathrm{II}} = \sqrt{K}dA$. Then by the Gauss-Bonnet theorem we have

$$\int K dA = \int K_{\mathrm{II}} dA_{\mathrm{II}} = \int K_{\mathrm{II}}\sqrt{K} dA = 4\pi$$

or

(4)
$$\int \sqrt{K}(\sqrt{K} - K_{\mathrm{II}}) dA = 0.$$

Proof of Theorem 2.1. For a critical point P_0 of H it follows from Lemma 2.2 that

$$cH^s(P_0)K^r(P_0) \geqslant H(P_0) \quad \text{or} \quad cH^s(P_0)H^{2r}(P_0) \geqslant H(P_0)$$

(since $H^{2r} \geqslant K^r$), and thus

$$cH^{s+2r-1}(P_0) \geqslant 1. \tag{5}$$

We distinguish two cases:

Case 1. Let $s+2r-1 \geqslant 0$. In this case we choose as P_0 a point such that

$$H(P_0) = \min_{P \in S} H(P)$$

(at least one such P_0 always exists, since S is closed and H is a continuous function). Then by (5) we have

$$cH^{s+2r-1} \geqslant 1 \quad \text{everywhere on } S. \tag{6}$$

Case 2. Let $s+2r-1 < 0$. In this case we choose as P_0 a point such that

$$H(P_0) = \sup_{P \in S} H(P).$$

Then by (5) we also obtain (6).

Now, from (6) we conclude that in every case we have $cH^{s+2r-1} \geqslant 1$ or

$$cH^s \geqslant \frac{1}{H^{2r-1}} \quad \text{everywhere on } S. \tag{7}$$

But $0 \leqslant r \leqslant \frac{1}{2}$ or $2r-1 \leqslant 0$, and thus

$$H^{2r-1} \leqslant K^{r-1/2} \quad \text{or} \quad \frac{1}{H^{2r-1}} \geqslant \frac{1}{K^{r-1/2}}.$$

Then by (7) we obtain $cH^s \geqslant 1/K^{r-1/2}$ or $cH^sK^r \geqslant \sqrt{K}$, or $K_{\mathrm{II}} \geqslant \sqrt{K}$ everywhere on S. Finally, from (4) we get $K_{\mathrm{II}} = \sqrt{K}$, and by a theorem in [4], p. 177, we conclude that S is a sphere.

3. It is well known (see [4], p. 177) that an ovaloid in E^3 with $K_{\mathrm{II}} = \sqrt{K}$ is a sphere. In this section we give a theorem which contains in some way a generalization of the above result.

THEOREM 3.1. *Let S be a complete and convex surface in E^3 with $K_{\mathrm{II}} \leqslant \sqrt{K}$. If the function H^2/K attains its maximum, then S is a sphere.*

Proof. Let $P_0 \in S$ be a point at which the function H^2/K attains its maximum. By Lemma 2.1 we have $\sqrt{K}(P_0) \geqslant K_{\mathrm{II}}(P_0) = H(P_0)$, which means that P_0 is an umbilic point, and thus $(H^2/K)(P_0) = 1$. Since at P_0 the function H^2/K attains its maximum, we have $(H^2/K)(P) = 1$ for all $P \in S$, i.e. all points of S are umbilics. Since S is convex and complete, it must be a sphere.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOANNINA
IOANNINA

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