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UNIFORMIZABLE $\Lambda(2)$ SETS AND UNIFORM INTEGRABILITY

BY

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Given a subset E of the integers, let L_E^2 be the space of all functions f in $L^2(T)$ whose Fourier coefficients $\hat{f}(n)$ vanish off the set E. It is shown that E is a uniformizable $\Lambda(2)$ set if and only if the functions $|f|^2$ for f in the unit ball of L_E^2 form a uniformly integrable family. It follows that if E is a uniformizable $\Lambda(2)$ set, then there is an Orlicz space $L^{\Phi}(T)$ strictly smaller than $L^2(T)$ such that $L_E^2 \subset L^{\Phi}(T)$. Some other characterizations of uniformizability of $\Lambda(2)$ sets are also given. Finally, the relationship between this property and 2-associatedness is explored.

1. Uniformizability. We review some of the notions considered here; see [6] for more details. Our results hold for all compact abelian groups, but we state them here for the unit circle T. Call an integrable function f on T an E-*function* if $\hat{f}$ vanishes on $Z \setminus E$, the complement of E in the set Z of all integers. Call f an E-*polynomial* if it is a trigonometric polynomial that is an E-function. Call E a $\Lambda(2)$ *set* if there is a constant $c(E)$ so that $\|f\|_2 \leqslant c(E)\|f\|_1$ for all E-polynomials f. An alternate characterization of $\Lambda(2)$-ness is that every E-function, which a priori lies in $L^1(T)$, belongs to $L^2(T)$. We need a dual characterization [11, § 5.3]: A set E is a $\Lambda(2)$ set if and only if there is a constant $c(E)$ so that for each function v in $l^2(E)$ there is a continuous function g with

(i) $$\|g\|_\infty \leqslant c(E)\|v\|_2,$$

(ii) $$\hat{g}(n) = v(n) \quad \text{for all } n \text{ in } E.$$

There is no direct requirement here on $\hat{g}|(Z \setminus E)$, the restriction of $\hat{g}$ to the complement of E; note, however, that it follows from conditions (i) and (ii)

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that

$$\text{(1)} \qquad \|\hat{g}|(Z\setminus E)\|_2 = [(\|g\|_2)^2-(\|v\|_2)^2]^{1/2} \leqslant [c(E)^2-1]^{1/2}\|v\|_2.$$

Call E a *uniformizable* $\Lambda(2)$ *set* if for each number $\varepsilon > 0$ there is a constant $c(E, \varepsilon)$ so that for each function v in $l^2(E)$ there is a continuous function g satisfying conditions (i) and (ii) above, with $c(E)$ replaced by $c(E, \varepsilon)$, and also satisfying the condition

$$\text{(iii)} \qquad \|\hat{g}|(Z\setminus E)\|_2 \leqslant \varepsilon\|v\|_2.$$

This notion was introduced by R. C. Blei [2]. Every set that is known to be a $\Lambda(2)$ set is known [2, Lemma 2.2] to be uniformizable; moreover, it is easy to see that the union of a $\Lambda(2)$ set and a uniformizable $\Lambda(2)$ set is a $\Lambda(2)$ set. This connection with the, as yet unsolved, union problem for $\Lambda(2)$ sets is one of the main reasons for studying uniformizability of $\Lambda(2)$ sets, but the notion was in fact introduced [2] for other purposes. The present paper is concerned with the connection between this property and other more familiar properties of thin sets.

Identify the unit circle T with the interval $[0, 2\pi)$, carrying the measure $dt/2\pi$; given a measurable subset S of T, denote the measure of S by $|S|$. Recall that a family $\mathscr{F}$ of functions on T is called *uniformly integrable* [10, § 3.1] if for each number $\varepsilon > 0$ there is a number $\delta > 0$ such that $\int_S |f| < \varepsilon$ for all measurable sets S with $|S| < \delta$. Finally, see [5] for basic facts about Young functions and Orlicz spaces; we use the Luxemberg norm on Orlicz spaces, so that some of the inequalities that we write differ by a factor of 2 from those in [5].

Given a subset E of Z, let $\mathscr{F}_E$ be the family of all functions $|f|^2$ with f in the unit ball of L^2_E.

THEOREM 1. *The following properties of a set E of integers are equivalent*:

(a) *E is a uniformizable* $\Lambda(2)$ *set.*

(b) *The family* $\mathscr{F}_E$ *is uniformly integrable.*

(c) *There is a Young function* Φ *with the property that* $\Phi(x)/x^2 \to \infty$ *as* $x \to \infty$ *for which* $L^2_E \subset L^\Phi(T)$.

Proof. We need another characterization of uniformizability.

LEMMA. *In order that E be a uniformizable* $\Lambda(2)$ *set, it is necessary and sufficient that for each number* $\varepsilon > 0$ *there exist a constant* $C(E, \varepsilon)$ *such that for each function f in* L^2_E *there is a function g in* $L^\infty(T)$ *satisfying the conditions*:

$$\text{(I)} \qquad \|g\|_\infty \leqslant C(E, \varepsilon)\|f\|_2,$$

$$\text{(II)} \qquad \|g-f\|_2 \leqslant \varepsilon\|f\|_2.$$

Although we will not use this fact in the sequel, we remark that there is no mystery about the best choice of the function g, given the function f and the constant C; to minimize $\|g-f\|_2$ among all functions g with $\|g\|_\infty \leqslant C\|f\|_2 = M$ say, let $g = f$ on the set where $|f| \leqslant M$, and let $g = (\operatorname{sgn} f)\cdot M$ elsewhere.

We first show how the theorem follows from the Lemma, and then we prove the Lemma. We suppose first that E has property (a). Given a function f in $\mathscr{F}_E$ and a number $\varepsilon > 0$, we apply the Lemma with ε replaced by $\sqrt{\varepsilon}/2)$ to get a function g so that $\|g\|_\infty \leqslant C = C(E, \sqrt{\varepsilon}/2)$ and $\|g-f\|_2 \leqslant \sqrt{\varepsilon}/2$. Then for each measurable subset S of T, we have that

$$\Big(\int_S |f|^2\Big)^{1/2} \leqslant \Big(\int_S |g|^2\Big)^{1/2} + \sqrt{\varepsilon}/2.$$

So, the requirements for uniform integrability of the family $\mathscr{F}_E$ are clearly satisfied with $\delta = \varepsilon/(4C^2)$.

Suppose next that E has property (b). Let ε and f be as in the Lemma. Suppose without loss of generality that $\|f\|_2 = 1$. For a value of C to be specified later, let S be the set where $|f| > C$. Then let $g = 0$ on S, and let $g = f$ elsewhere. Certainly, condition (I) holds. Also,

$$\|g-f\|_2 = \Big(\int_S |f|^2\Big)^{1/2}.$$

The uniform integrability of $\mathscr{F}_E$ guarantees that the quantity above is at most ε provided that $|S| \leqslant \delta$ for a suitable positive constant $\delta(\varepsilon^2)$. On the other hand, by Chebyshev's inequality, $|S| < 1/C^2$. Now specify that $C = 1/\sqrt{\delta}$. Then condition (II) is satisfied, and by the Lemma, E is a uniformizable $\Lambda(2)$ set.

We deal next with the equivalence of conditions (b) and (c). This is essentially known [10, § 3.1], in a slightly different formulation. Suppose first that the family $\mathscr{F}_E$ is uniformly integrable. By [10, Theorem 3.1.2] there is a "strongly convex" function ϕ and a positive constant M so that

$$\int \phi(|f|^2) \leqslant M$$

for all functions f in the unit ball of L^2_E. Let Φ be a Young function with the property that $\Phi(x) = \phi(x^2)$ for all sufficiently large values of x. Then the inequality above guarantees that $L^2_E \subset L^\Phi(T)$; moreover, the "strong convexity" of ϕ implies that $\Phi(x)/x^2 \to \infty$ as $x \to \infty$.

Suppose, on the other hand, that the set E has property (c). Since convergence in L^2_E and in $L^\Phi(T)$ each imply convergence in measure, the closed-graph theorem guarantees that the inclusion $L^2_E \to L^\Phi(T)$ is continuous. Thus there is a constant M so that $\int \Phi(|f|/M) \leqslant 1$ for all functions f in the unit ball, B_E say, of L^2_E. The argument in [10, § 3.1] then shows that the family of all functions $(|f|/M)^2$ as f runs through B_E is uniformly integrable. Therefore the family $\mathscr{F}_E$ is also uniformly integrable. This completes the proof of the theorem.

We now prove the Lemma. Suppose first that E is a uniformizable $\Lambda(2)$ set. Fix a number $\varepsilon > 0$. Let $f \in L^2_E$, and let v be the restriction of $\hat{f}$ to E. By the uniformizability of E, there is a continuous function g satisfying conditions (i), (ii), and (iii) in the definition of uniformizable $\Lambda(2)$ set. In particular, the fact that $\hat{g}|E = \hat{f}|E$ implies that

$$\begin{aligned}\|\hat{g}-\hat{f}\|_2 &= \|(\hat{g}-\hat{f})|(Z\setminus E)\|_2\\ &= \|\hat{g}|(Z\setminus E)\|_2, \quad \text{because } f \text{ is an } E\text{-function,}\\ &\leqslant \varepsilon\|\hat{f}\|_2.\end{aligned}$$

Hence $\|g-f\|_2 \leqslant \varepsilon\|f\|_2$ as required. Moreover, condition (II) holds with $C = c(E, \varepsilon)$.

Conversely, suppose that E satisfies the conditions specified in the Lemma. Fix a positive number ε and a function v in $l^2(E)$. Let f_1 be the function in L^2_E with $\hat{f}_1|E = v$. Apply the conditions in the Lemma with ε replaced by $\varepsilon/4$ to get a function, h_1 say, in $L^\infty(T)$ with the properties:

(I′) $$\|h_1\|_\infty \leqslant C(E, \varepsilon/4)\|f_1\|_2,$$

(II′) $$\|h_1-f_1\|_2 \leqslant (\varepsilon/4)\|f_1\|_2.$$

Choose a trigonometric polynomial, P_1 say, with $\|P_1\|_\infty \leqslant \|h_1\|_\infty$, and $\|P_1-h_1\|_2 \leqslant (\varepsilon/4)\|f_1\|_2$. Then

$$\|P_1\|_\infty \leqslant C(E, \varepsilon/4)\|f_1\|_2,$$

$$\|P_1-f_1\|_2 \leqslant (\varepsilon/2)\|f_1\|_2.$$

Let f_2 be the function in L^2_E with $\hat{f}_2 = \hat{f}_1 - \hat{P}_1$ on E. Then $\|f_2\|_2 \leqslant (\varepsilon/2)\|f_1\|_2$. As above, choose a trigonometric polynomial P_2 so that $\|P_2\|_\infty \leqslant C(E, \varepsilon/4)\|f_2\|_2$, and $\|P_2-f_2\|_2 \leqslant (\varepsilon/2)\|f_2\|_2$. Let f_3 be the function in L^2_E with $\hat{f}_3 = \hat{f}_2 - \hat{P}_2$ on E, and continue in this fashion to obtain a sequence $(P_n)_{n=1}^\infty$ of trigonometric polynomials such that

$$\|P_n\|_\infty \leqslant C(E, \varepsilon/4)(\varepsilon/2)^{n-1}\|v\|_2,$$

and such that

(2) $$\Big\|\Big(\hat{f}_1|E - \sum_{j=1}^{n}\hat{P}_j|E\Big)\Big\|_2 \leqslant (\varepsilon/2)^n\|v\|_2.$$

There is no loss of generality in assuming that $\varepsilon \leqslant 1$. In that case, the series $\sum_{n=1}^{\infty} P_n$ converges uniformly to a continuous function, g say, with

$$\|g\|_\infty \leqslant \frac{C(E, \varepsilon/4)}{(1-\varepsilon/2)}\|v\|_2.$$

It follows from inequality (2) above that $\hat{g}|E = \hat{f}_1|E = v$. Finally,

$$\|\hat{g}|(Z\setminus E)\|_2 \leqslant \sum_{n=1}^{\infty} \|f_n - P_n\|_2 \leqslant \sum_{n=1}^{\infty} (\varepsilon/2)^n \|v\|_2$$
$$= \frac{\varepsilon/2}{1-\varepsilon/2}\|v\|_2,$$

which is at most $\varepsilon\|v\|_2$, because $\varepsilon \leqslant 1$ here. This shows that E is a uniformizable $\Lambda(2)$ set, with $c(E, \varepsilon) \leqslant C(E, \varepsilon/4)/(1-\varepsilon/2)$.

Remarks. 1. In an announcement [John J. F. Fournier, Abstracts of Papers Presented to the American Mathematical Society 4 (1983), # 805–43–86] of these and other results, it was erroneously stated that E is a uniformizable $\Lambda(2)$ set if and only if

(d) There is an Orlicz space $L^{\Phi}(T)$ strictly smaller than $L^2(T)$ such that $L^2_E \subset L^{\Phi}(T)$.

In fact, it does not seem to be known whether this equivalence holds. The condition that $\Phi(x)/x^2 \to \infty$ as $x \to \infty$ does imply that the inclusion $L^{\Phi}(T) \subset L^2(T)$ is strict; so, by part (c) of Theorem 1, uniformizability implies condition (d). There are examples, however, of Young functions Φ for which

$$0 < \liminf_{x\to\infty} \Phi(x)/x^2 < \limsup_{x\to\infty} \Phi(x)/x^2 = \infty.$$

Then $L^{\Phi}(T)$ is strictly included in $L^2(T)$, but it is not clear whether the inclusion $L^2_E \subset L^{\Phi}(T)$ for such a Young function Φ implies that E is uniformizable, or even that E is a $\Lambda(2)$ set.

2. G. Pisier observed in [2, Lemma 2.2] that if E is a $\Lambda(q)$ set for some index $q > 2$, then E is a uniformizable $\Lambda(2)$ set. This is a special case of the implication (c) $\Rightarrow$ (a) above, while the implication (a) $\Rightarrow$ (c) is a partial converse to Pisier's observation.

3. As noted in the introduction, E is a $\Lambda(2)$ set if and only if $L^1_E \subset L^2(T)$. On the other hand, it follows easily from Theorem 1 that E is a uniformizable $\Lambda(2)$ set if any only if $L^1_E \subset L^{\Phi}(T)$ for some Young function Φ with the property that $\Phi(x)/x^2 \to \infty$ as $x \to \infty$.

4. A Young function has the property above if and only if [5, Exercise 3.5.13] the complementary Young function, Ψ say, has the property that

$$x^2/\Psi(x) \to \infty \quad \text{as } x \to \infty. \tag{3}$$

By a standard duality argument [6, 5.3 (vi)], E is a uniformizable $\Lambda(2)$ set if any only if there is a Young function Ψ with property (3) for which $\hat{f}|E \in l^2(E)$ for all functions f in $L^{\Psi}(T)$.

5. Observe that if $\varepsilon \geqslant 1$, then conditions (I) and (II) of the Lemma are trivially satisfied with $g = 0$. Given a subset E of Z, let $S(E)$ be the set of all numbers ε for which for each function f in L^2_E there is a function g in $L^{\infty}(T)$

satisfying conditions (I) and (II). Let $\varepsilon(E) = \inf S(E)$. Then $0 \leqslant \varepsilon(E) \leqslant 1$ for all sets E, and, by the Lemma, E is a uniformizable $\Lambda(2)$ set if and only if $\varepsilon(E) = 0$. It is interesting that

$$E \text{ is a } \Lambda(2) \text{ set} \quad \Leftrightarrow \quad \varepsilon(E) < 1. \tag{4}$$

This characterization of $\Lambda(2)$-ness seems to be new, although an analogous condition is known [1] to be necessary for E to be a $\Lambda(1)$ set. We omit the proof of assertion (4), but we note that it may be useful in dealing with the union problem for $\Lambda(2)$ sets. Indeed, it easy to show that

$$\varepsilon(E \cup F) \leqslant [\varepsilon(E)^2 + \varepsilon(F)^2]^{1/2},$$

for all pairs of sets E and F. It is immediate that if E is a uniformizable $\Lambda(2)$ set, and F is a $\Lambda(2)$ set, then $E \cup F$ is a $\Lambda(2)$ set. Even if neither E nor F is uniformizable, their union is a $\Lambda(2)$ set if $\varepsilon(E)$ and $\varepsilon(F)$ are small enough. The family of $\Lambda(2)$ sets is usually parametrized by the *$\Lambda(2)$-constant*, which for a given set E is defined to be the infimum, $c(E)$ say, of the numbers C for which it is true that $\|f\|_2 \leqslant C\|f\|_1$ for all E-polynomials f. It can be shown that $\varepsilon(E) \leqslant [1 - 1/c(E)^2]^{1/2}$. It follows that if $c(E)^{-2} + c(F)^{-2} > 1$, then $E \cup F$ is a $\Lambda(2)$ set. In order for this inequality to hold, however, it is necessary that at least one of $c(E)$ and $c(F)$ be strictly less than $\sqrt{2}$. Unfortunately, most $\Lambda(2)$ sets have constants much larger than $\sqrt{2}$.

6. The parameter $\varepsilon(E)$ defined above also has a dual characterization. Consider inequalities of the form

$$\|P\|_2 \leqslant C\|P+Q\|_1 + \eta\|P+Q\|_2, \tag{5}$$

where P is an E-polynomial, Q is a $(Z \setminus E)$-polynomial, and C and η are positive constants. Then $\varepsilon(E)$ is the infimum of the set of numbers η for which there is a constant C such that inequality (5) holds for all such polynomials P and Q.

2. Two-associatedness. Next we investigate the connection between uniformizability and 2-associatedness. Let E be a subset of Z, and S be a measurable subset of T; denote the indicator function of S by 1_S. Recall [6] that E is said to be *strictly 2-associated with S* if there is a constant $\varkappa$ so that $\|P\|_2 \leqslant \varkappa\|1_S \cdot P\|_2$ for all E-polynomials P. Also, E is said to be *2-associated with S* if E has a finite subset, F say, so that $E \setminus F$ is strictly 2-associated with S. Finally, we say that E *tends to infinity* if the difference between successive elements of E tends to infinity.

THEOREM 2. *Suppose that E is a uniformizable $\Lambda(2)$ set, and that E is strictly 2-associated with S; then there is a number $\delta > 0$ so that E is strictly 2-associated with every measurable set S_1 for which $|S \setminus S_1| < \delta$. Suppose* *further that E tends to infinity; then E is 2-associated with every measurable set of positive measure.*

Proof. Let E be uniformizable and strictly 2-associated with S. Then there is a constant $\varkappa$ so that

$$\int_S |P|^2 \geqslant 1/\varkappa^2$$

for all E-polynomials P with $\|P\|_2 = 1$. Since the family $\mathscr{F}_E$ is uniformly integrable, there is a positive number δ so that if R is a measurable set with $|R| < \delta$, then $\int_R |P|^2 \leqslant 1/2\varkappa^2$ for all such functions P. Suppose that $|S \setminus S_1| < \delta$; then, for all such P,

$$\int_{S_1} |P|^2 \geqslant \int_S |P|^2 - \int_{S \setminus S_1} |P|^2 \geqslant \frac{1}{\varkappa^2} - \frac{1}{2\varkappa^2} = \frac{1}{2\varkappa^2}.$$

So, E is strictly 2-associated with S_1.

Let us say that two sets E and S are *very strongly* 2-*associated* if for each number $\lambda > 1$ there is a finite set F so that

$$\frac{1}{\lambda}(\|f\|_2)^2 \leqslant \frac{1}{|S|}\int_S |f(t)|^2 \frac{dt}{2\pi} \leqslant \lambda(\|f\|_2)^2$$

for all $(E \setminus F)$-polynomials f; in other words, for all such functions f, the average of $|f|^2$ over the set S is nearly equal to its average over all of T. It was shown by Zygmund [9, Theorem V.6.10] that the classical lacunary sets are very strongly 2-associated with every set of positive measure; the same argument [3, p. 396] shows that every $\Lambda(4)$ set that tends to infinity has this property.

We now use Theorem 1 to show that every uniformizable $\Lambda(2)$ set that tends to infinity is very strongly 2-associated with every set of positive measure. To this end, fix such a subset E of Z and a measurable set S of positive measure. For a value of N to be specified later, let P_N be the Fejer mean of order N of the Fourier series of 1_S. Given N, choose a finite set F so that successive terms in the set $E \setminus F$ all differ by more than N. Then $(P_N \cdot f)\hat{}(n) = \hat{P}_N(0)\hat{f}(n) = |S|\hat{f}(n)$ for all $(E \setminus F)$-functions f and all integers n in $(E \setminus F)$. So,

$$\frac{1}{|S|}\int_T P_N |f|^2 = \frac{1}{|S|}\sum_{n \in E \setminus F} |(P_N \cdot f)\hat{}(n) \cdot \hat{f}(n)| = (\|f\|_2)^2 \tag{6}$$

for all such functions f.

Our goal is to show that $\frac{1}{|S|}\int\limits_T 1_S\cdot|f|^2$ is nearly equal to $(\|f\|_2)^2$. By formula (6), we know that the latter quantity is exactly equal to $\frac{1}{|S|}\int\limits_T P_N\cdot|f|^2$. So, it suffices to show, given $\varepsilon>0$, that N can be chosen so that

$$\Big|\int\limits_T P_N\cdot|f|^2-\int\limits_T 1_S\cdot|f|^2\Big|\leqslant\varepsilon|S|(\|f\|_2)^2$$

for all E-functions f. To this end, recall that in the proof of Theorem 1, the Young function Φ in part (c) had the form Φ: $x\to\phi(x^2)$, where ϕ was a suitable "strongly convex" function. Suppose, without loss of generality that ϕ itself is a Young function, and thus has a conjugate, ψ say. Define a Young function B by letting $B(x)=\psi(x^2)$ for all x. Since $\int|FG|\leqslant 2$ for all functions F in the unit ball of $L^\varphi(T)$ and G in the unit ball of $L^\psi(T)$, it is also true that $\int|fg|^2\leqslant 2$ for all f in the unit ball of $L^\Phi(T)$ and all g in the unit ball of $L^B(T)$. Thus,

$$\|fg\|_2\leqslant\sqrt{2}\,\|f\|_\Phi\|g\|_B$$

for all measurable functions f and g.

Recall also that the imbedding $L^2_E\to L^\Phi(T)$ is continuous, with norm M say. Since the function 1_S is bounded, it can be approximated arbitrarily-closely in the L^B-norm by continuous functions: it follows that $\|P_N-1_S\|_B\to 0$ as $N\to\infty$. Given a number $\varepsilon>0$, choose N so that $\|P_N-1_S\|_B\leqslant\varepsilon|S|/(\sqrt{2}M)$. Then,

$$\begin{aligned}\Big|\int\limits_T 1_S\cdot|f|^2-\int\limits_T P_N\cdot|f|^2\Big| &\leqslant\int\limits_T|1_S-P_N||f|^2\\ &\leqslant\|(1_S-P_N)\cdot f\|_2\|f\|_2\\ &\leqslant\sqrt{2}\,\|1_S-P_N\|_B\|f\|_\Phi\|f\|_2\\ &\leqslant\sqrt{2}(\varepsilon|S|/(\sqrt{2}M))(M\|f\|_2)\|f\|_2\\ &\leqslant\varepsilon|S|(\|f\|_2)^2,\end{aligned}$$

as required.

Remarks. 7. As noted at the beginning of the paper, our results hold, with the same proofs, in the context of compact abelian groups. See [6] for the appropriate extension to this context of the condition that E tend to infinity. Some such hypothesis is needed in the second part of Theorem 2 in order for that part of the theorem to be valid for all compact abelian groups, but it is not known whether such a condition is necessary on T and other connected groups. Also, on connected groups, the argument in [3, Theorem

IV.4] and the first part of Theorem 2 show that if a uniformizable set E is 2-associated with a set S, then it is *strictly* 2-associated with S; so, the conclusion of the second part of the theorem can be strengthened in another way.

8. The theorem above is motivated by some observations of Bonami [3]. She proved the first conclusion of Theorem 2 under the assumption that E is a $\Lambda(q)$ set for some index $q > 2$, and the second conclusion of the theorem under the assumption that E is a $\Lambda(4)$ set that tends to infinity. On the other hand, the results in the present paper were obtained before the author learned about the work of Miheev [9, 8, 7], which appeared after Bonami's paper, and which includes results that, in some cases, are stronger in one important respect than the second part of Theorem 2. It is shown by Miheev in [8] that if a subset E of the integers is a $\Lambda(p)$ set for some index $p > 2$, then E is 2-associated with every measurable set of positive measure in the circle group. There is no requirement in this theorem that E tend to infinity! As noted in Remark 2 of the previous section, every set that is $\Lambda(p)$ for some $p > 2$ is a uniformizable $\Lambda(2)$ set, but it is not known whether the converse holds, so that our theorem may apply to some sets not covered by Miheev's theorem. Every set that is known to be a $\Lambda(2)$ set is, however, known to be a $\Lambda(4)$ set too, so this weakening of hypothesis may be purely formal. It is not clear to what extent Miheev's methods also work for general compact abelian groups; as noted above, some version of the hypothesis that E tend to infinity is definitely needed for some groups. My student Kathryn Hare has recently used some of the methods in [6, Chapter 8] to show, for any compact abelian group, that if a uniformizable $\Lambda(2)$ set, E say, is a union of finitely-many sets that tend to infinity, and if E is "almost X_0-transversal" [6, 8.2] for all finite subgroups X_0 of the dual group, then E is 2-associated with every set of positive measure.

9. Our hypothesis that E tend to infinity is necessary for our conclusion that E be very strongly 2-associated with every set of positive measure. To see this, let S be any measurable set whose complement has positive measure. Since the indicator function of S^c can be approximated arbitrarily-closely in the L^2-norm by trigonometric polynomials, there is, for each number C, such a polynomial P with $\|P\|_2 > C\|P\cdot 1_S\|_2$. Let F be the finite set of integers where $\hat{P} \neq 0$, let D be an infinite Sidon set, and let E be the algebraic sum $F+D$. Then E is a Sidon set, and a fortiori a $\Lambda(p)$ set for all $p < \infty$. Let E' be any set obtained from E by deleting a finite subset. Then for most integers n in D it is the case that $F+n \subset E'$. For any such integer n the polynomial P_n given by $P_n(t) = e^{int}P(t)$ has the property that $\|P_n\|_2 > C\|P_n\cdot 1_S\|_2$. To summarize, for each set S as above, there is a Sidon set E so that E and S fail to be very strongly 2-associated.

10. Say that E is *strictly* (2, 1)*-associated with* S if there is a constant $\varkappa$ so that $\|P\|_2 \leqslant \varkappa\|1_S\cdot P\|_1$ for all E-polynomials. There is an alternate proof

of the second part of Theorem 2, based on the definition of uniformizable $\Lambda(2)$ set rather than on Theorem 1, that produces a finite set F for which $E \setminus F$ is strictly (2, 1)-associated with S_1. In fact, however, if a uniformizable $\Lambda(2)$ set is strictly 2-associated with S, then the set must be strictly (2, 1)-associated with S. One way to prove this is to combine the first part of Theorem 2 above with the observation made in [4, Remark 4] that if there is a number $\delta > 0$ for which a given uniformizable $\Lambda(2)$ set E is strictly 2-associated with every measurable set S_1 for which $|S \setminus S_1| < \delta$, then E is strictly (2, 1)-associated with S.

11. Fix a T^* summability method as in [12, § III.1]. The notion of 2-associatedness is useful in showing [12, § V.6] that the classical lacunary trigonometric series have the property that if such a series is bounded T^* at almost every point on the circle, then the series is square-summable. It is easy to see that if this implication holds for all series with frequencies in a given set E, then E must be a $\Lambda(2)$ set. Conversely, if E is a $\Lambda(2)$ set, and a series with frequencies only in E is almost-everywhere bounded T^*, then the series is square-summable. This follows from a result in [4, Theorem 1], asserting that E is a $\Lambda(2)$ set if and only if there is number $\varepsilon > 0$ so that E is strictly 2-associated with every measurable set of measure greater than $1-\varepsilon$.

12. There are situations [12, V.6.10] where, for real-valued lacunary series, one-sided boundedness T^* on a set of positive measure implies that the series is square-summable. A glance at the argument in [12] shows that this implication holds for such series with frequencies only in $E \cup (-E)$ provided that $E \cup (-E)$ is (2, 1)-associated with every set of positive measure. A sufficient condition for the latter implication is that E be a uniformizable $\Lambda(2)$ set that tends to infinity.

Added in proof. Some of the questions mentioned in Remarks 7 and 8 have been answered by Kathryn E. Hare in her Thesis *Thin sets and strict two-associatedness.*

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