COLLOQUIUM MATHEMATICUM

VOL. LV 1988 FASC. 1

SIMPLE CONFORMALLY SYMMETRIC MANIFOLDS WITH METRIC SEMI-SYMMETRIC CONNECTION

BY

J. KRAWCZYK (OPOLE)

1. Introduction. Let M be a Riemannian manifold with a (possibly indefinite) metric g. According to Adati and Miyazawa [1], an n-dimensional ($n \geqslant 4$) Riemannian manifold M is called *conformally recurrent* if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \tag{1}$$

is recurrent, i.e.,

$$\nabla_l C_{hijk} = a_l C_{hijk}. \tag{2}$$

Investigating conformally recurrent manifolds, Roter has introduced the concept of a simple conformally recurrent manifold (s.c.r. in short) and proved the following results:

THEOREM A (see [6], Theorem 1). *A Riemannian manifold* (M, g) *of dimension* $n \geqslant 4$ *is s.c.r. if and only if*

(i) (M, g) *is not conformally flat,*

(ii) $\nabla_l C_{hijk} = a_l C_{hijk}$,

(iii) *the recurrence vector* a_j *is locally a gradient,*

(iv) *the Ricci tensor is a Codazzi one.*

THEOREM B (see [6], Theorem 5). *Let M be a non-locally symmetric s.c.r. manifold. If M is not Ricci-recurrent, then*

$$C_{hijk} = e\omega_{hi}\omega_{jk}, \tag{A}$$

where $|e| = 1$ *and* ω *is a uniquely determined recurrent absolute 2-form satisfying* $\operatorname{rank}\omega = 2$ *and* $\omega_{ir}\omega^r{}_j = 0$.

THEOREM C (see [6], Lemma 8). *Let M be a non-locally symmetric s.c.r. manifold such that*

$$d_i C_{hljk} + d_k C_{hlij} + d_j C_{hlki} = 0$$

for some field d_j of non-zero vectors. If C_{hijk} is not of the form (A), *then $d_{i,j} = A_j d_i$ for a certain vector field A_j on M. Moreover, if $d_{i,j} = d_{j,i}$, then* rank $R_{ij} \leqslant 1$.

Suppose that a Riemannian manifold M with metric g_{ij} admits a metric semi-symmetric connection with connection coefficients $\mathring{\Gamma}^h_{ji}$ given by (see [3])

$$\mathring{\Gamma}^h_{ji} = \Gamma^h_{ji} + \delta^h_j p_i - p^h g_{ji}, \tag{3}$$

where Γ^h_{ji} are Christoffel symbols of M, p_i is a gradient field such that $p_i \neq 0$ at least at one point $x \in M$, and $p^h = g^{hr} p_r$. In this paper $\mathring{\nabla}$ denotes covariant differentiation with respect to $\mathring{\Gamma}^h_{ji}$. The curvature tensors $\mathring{R}^k{}_{jih}$ of $\mathring{\Gamma}^h_{ji}$ and $R^k{}_{jih}$ of Γ^h_{ji} are related by

$$\mathring{R}_{kjih} = R_{kjih} - \alpha_{ji} g_{kh} + \alpha_{ki} g_{jh} - \alpha_{kh} g_{ji} + \alpha_{jh} g_{ki},$$

where α_{ji} is a tensor field of type (0, 2) defined by

$$\alpha_{ij} = \nabla_j p_i - p_i p_j + \tfrac{1}{2} p_r p^r g_{ij}, \qquad \mathring{R}_{kjih} = g_{kr} \mathring{R}^r{}_{jih}.$$

Let $\mathring{C}_{kjih}$ denote the conformal curvature tensor relative to the metric semi-symmetric connection, i.e.,

$$\mathring{C}_{kjih} = \mathring{R}_{kjih} - \frac{1}{n-2}(g_{kh} \mathring{R}_{ji} - g_{jh} \mathring{R}_{ki} + g_{ji} \mathring{R}_{kh} - g_{ki} \mathring{R}_{jh}) + \frac{\mathring{R}}{(n-1)(n-2)}(g_{kh} g_{ji} - g_{jh} g_{ki}),$$

where $\mathring{R}_{kh} = g^{ji} \mathring{R}_{kjih}$ and $\mathring{R} = g^{kh} \mathring{R}_{kh}$.

An n-dimensional ($n \geqslant 4$) Riemannian manifold is called *conformally recurrent with respect to $\mathring{\nabla}$* if its conformal curvature tensor $\mathring{C}_{kjih}$ satisfies the condition

$$\mathring{\nabla}_l \mathring{C}_{kjih} = \mathring{a}_l \mathring{C}_{kjih}. \tag{4}$$

If $\mathring{\nabla}_l \mathring{C}_{kjih} = 0$ everywhere on M and $\dim M \geqslant 4$, then M is called *conformally symmetric with respect to the metric semi-symmetric connection.*

Investigating Riemannian manifolds satisfying (2) and (4), the author has proved the following results:

THEOREM D (see [4], Theorem 3). *Let M be a Riemannian manifold which admits a metric semi-symmetric connection* (3) *such that conditions* (2) *and* (4) *hold. Then*

(a) $p_l C_{hijk} + p_k C_{hilj} + p_j C_{hikl} = 0$ *everywhere on M.*

(b) *If (M, g) is not conformally flat, then $\mathring{a}_j = a_j - 2p_j$ and $p_r p^r = 0$.*

THEOREM E (see [4], Theorem 4). *Let M be a Riemannian manifold which admits a metric semi-symmetric connection* (3) *such that the function p satisfies equation* (a). *Then*

(c) $\mathring{\nabla}_l \mathring{C}_{hijk} = \nabla_l C_{hijk} - 2p_l C_{hijk}$.

An n-dimensional ($n \geqslant 4$) Riemannian manifold M is called *simple conformally symmetric with respect to the metric semi-symmetric connection* (s.c.s. in short) if M is not conformally flat and satisfies the conditions

$$\mathring{\nabla}_l \mathring{C}_{hijk} = 0, \qquad \nabla_l C_{hijk} = a_l C_{hijk}. \tag{5}$$

The present paper deals with some results on s.c.s. manifolds. It will be also shown that any s.c.s. manifold is necessarily s.c.r.

All Riemannian manifolds (possibly with indefinite metric) considered below are assumed to be connected and of class C^∞ or analytic.

2. Preliminaries. In the sequel we need the following results:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the well-known relations*

$$\begin{gathered} C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \\ C_{hijk} + C_{hkij} + C_{hjki} = 0, \qquad C^r{}_{ijr} = C^r{}_{irj} = C^r{}_{rij} = 0, \end{gathered} \tag{6}$$

$$\nabla_r C^r{}_{ijk} = \frac{n-3}{n-2}\left[\nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{ik}\nabla_j R)\right]. \tag{7}$$

LEMMA 2 (see [5], Lemma 3). *If c_j, φ_j and B_{hijk} are numbers satisfying*

$$c_l B_{hijk} + \varphi_h B_{lijk} + \varphi_i B_{hljk} + \varphi_j B_{hilk} + \varphi_k B_{hijl} = 0,$$

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \qquad B_{hijk} + B_{hkij} + B_{hjki} = 0,$$

then each $b_j = c_j + 2\varphi_j$ is zero or each B_{hijk} is zero.

LEMMA 3. *Let M be an s.c.s. manifold. Then*

$$H = C_{rijk} C^r{}_{hlm} + C_{hrjk} C^r{}_{ilm} + C_{hirk} C^r{}_{jlm} + C_{hijr} C^r{}_{klm} = 0. \tag{8}$$

Proof. Using (a), we can follow step by step a proof of Roter (see [6], Lemma 6) to obtain (8).

LEMMA 4. *Let M be an s.c.s. manifold. Then*

$$R_{mr} R^r{}_{ijk} + R_{kr} R^r{}_{imj} + R_{jr} R^r{}_{ikm} = 0, \tag{9}$$

$$R_{mr} C^r{}_{ijk} + R_{kr} C^r{}_{imj} + R_{jr} C^r{}_{ikm} = 0. \tag{10}$$

Proof. By Theorem E and (5), we get

$$\nabla_l C_{hijk} = 2p_l C_{hijk}. \tag{11}$$

Contracting (11) with g^{hl} we obtain $\nabla_r C^r{}_{ijk} = 2p_r C^r{}_{ijk}$. From (a), by transvection with g^{hl} and using (6), we find $p_r C^r{}_{ijk} = 0$. Thus $\nabla_r C^r{}_{ijk} = 0$. Substituting (7) into the above equation, we find

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-1)}(g_{ij} \nabla_k R - g_{ik} \nabla_j R),$$

whence, by covariant differentiation, we get

$$\nabla_l \nabla_k R_{ij} - \nabla_l \nabla_j R_{ik} = \frac{1}{2(n-1)}(g_{ij} \nabla_l \nabla_k R - g_{ik} \nabla_l \nabla_j R). \tag{12}$$

Permuting in (12) the indices j, k, l cyclically, adding the resulting equations to (12) and using the Ricci and Bianchi identities, we easily obtain (9). Relation (9), together with (1), leads immediately to (10). This completes the proof.

LEMMA 5. *Every s.c.s. manifold M satisfies the condition*

$$R_{mr} C^r{}_{ijk} = 0. \tag{13}$$

Proof. Since $\nabla_i p_j = \nabla_j p_i$, (11) implies

$$\nabla_l \nabla_m C_{hijk} - \nabla_m \nabla_l C_{hijk} = 0. \tag{14}$$

Applying to (14) the Ricci identity and using (1) and (8), we obtain

$$\begin{aligned} & R_{mr}(g_{hl} C^r{}_{ijk} + g_{il} C^r{}_{hkj} + g_{jl} C^r{}_{khi} + g_{kl} C^r{}_{jih}) \\ & \quad - R_{lr}(g_{hm} C^r{}_{ijk} + g_{im} C^r{}_{hkj} + g_{jm} C^r{}_{khi} + g_{km} C^r{}_{jih}) \\ & \quad + R_{hl} C_{mijk} + R_{il} C_{mhkj} + R_{jl} C_{mkhi} + R_{kl} C_{mjih} \\ & \quad - (R_{hm} C_{lijk} + R_{im} C_{lhkj} + R_{jm} C_{lkhi} + R_{km} C_{ljih}) \\ & = \frac{R}{n-1}[g_{hl} C_{mijk} + g_{il} C_{mhkj} + g_{jl} C_{mkhi} + g_{kl} C_{mjih} \\ & \quad - (g_{hm} C_{lijk} + g_{im} C_{lhkj} + g_{jm} C_{lkhi} + g_{km} C_{ljih})]. \end{aligned} \tag{15}$$

Contracting (15) with g^{hl} and using (6) and (10), we find

$$(n-1) R_{mr} C^r{}_{ijk} + R_{ir} C^r{}_{mjk} = g_{km} T_{ij} - g_{jm} T_{ik}, \tag{16}$$

where $T_{ij} = R^{rs} C_{rijs} = T_{ji}$. Symmetrizing (16) in m, i, we have

$$(n-1) R_{ir} C^r{}_{mjk} + R_{mr} C^r{}_{ijk} = g_{ki} T_{mj} - g_{ji} T_{mk}. \tag{17}$$

It follows from (16) and (17) that

$$R_{mr} C^r{}_{ijk} + R_{ir} C^r{}_{mjk} = \frac{1}{n}(g_{mk} T_{ij} - g_{ij} T_{mk} + g_{ik} T_{mj} - g_{mj} T_{ik}),$$

$$R_{mr} C^r{}_{ijk} - R_{ir} C^r{}_{mjk} = \frac{1}{n-2}(g_{mk} T_{ij} + g_{ij} T_{mk} - g_{ik} T_{mj} - g_{mj} T_{ik}).$$

The last two equations show that

(18) $$R_{mr} C^{r}{}_{ijk} = \frac{1}{n(n-2)} [(n-1)(g_{mk} T_{ij} - g_{mj} T_{ik}) + g_{ij} T_{mk} - g_{ik} T_{mj}].$$

Transvecting (18) with p^k and making use of $p^r T_{mr} = 0$, we get

$$(n-1) p_m T_{ij} = p_i T_{mj}.$$

Hence

$$p_i T_{mj} = (n-1) p_m T_{ij} = (n-1)^2 p_i T_{mj}$$

and, consequently, $T_{ij} = 0$. The assertion follows now from (18).

3. **Main results.** We are now going to derive some consequences of the above results.

THEOREM 1. *Let M be an analytic s.c.s. manifold. Then $p_r R^r{}_i = 0$ and the scalar curvature of M vanishes.*

Proof. Transvecting (15) with p^m, using Theorem D and Lemma 5, we get

$$(n-1) p^r (R_{rh} C_{lijk} + R_{ri} C_{lhkj} + R_{rj} C_{lkhi} + R_{rk} C_{ljih}) = 2R p_l C_{hijk},$$

which, in view of Lemma 2, implies

(19) $$(n-1) p^r R_{rj} = R p_j.$$

Transvecting now (a) with R^l_p and using (13), we obtain

(20) $$p_r R^r{}_p = 0.$$

Substituting (20) into (19) we get $R = 0$. This completes the proof.

PROPOSITION. *Let M be an analytic s.c.s. manifold. If C_{hijk} is not of the form* (A), *then* rank $R_{ij} \leqslant 1$.

Proof. Using Theorem D, Theorem 1 and (15), we can follow step by step a proof of Derdziński and Roter (see [2], p. 14) to obtain the assertion.

THEOREM 2. *Every analytic s.c.s. manifold M is an s.c.r. manifold.*

Proof. It is easily seen that $C_{hijk} \neq 0$. From (c) we have

(21) $$\nabla_l C_{hijk} = 2p_l C_{hijk},$$

where $\nabla_i p_j = \nabla_j p_i$. Contracting now (21) with g^{hl} and using (a), we obtain $\nabla_r C^r{}_{ijk} = 0$. Substituting the last result into (7) and using Theorem 1, we obtain $\nabla_k R_{ij} = \nabla_j R_{ik}$. This means that the Ricci tensor is a Codazzi tensor. Thus Theorem A completes the proof.

REFERENCES

[1] T. Adati and T. Miyazawa, *On a Riemannian space with recurrent conformal curvature*, Tensor (N.S.) 18 (1967), pp. 348–354.

[2] A. Derdziński and W. Roter, *Some properties of conformally symmetric manifolds which are not Ricci-recurrent*, ibidem 34 (1980), pp. 11–20.

[3] T. Imai, *Riemannian manifold satisfying a certain curvature condition*, ibidem 34 (1980), pp. 50–52.

[4] J. Krawczyk, *On some Riemannian manifolds admitting a metric semi-symmetric connection*, this fasc., pp. 55–59.

[5] W. Roter, *On conformally related conformally recurrent metrics, I. Some general results*, Colloq. Math. 47 (1982), pp. 39–46.

[6] – *On a class of conformally recurrent manifolds*, Tensor (N.S.) 39 (1982), pp. 207–217.

PEDAGOGICAL UNIVERSITY
OPOLE

Reçu par la Rédaction le 22.2.1985