COLLOQUIUM MATHEMATICUM

VOL. XXIV 1972 FASC. 2

METRIZABILITY OF INVERSE IMAGES OF METRIC SPACES UNDER OPEN PERFECT AND 0-DIMENSIONAL MAPPINGS

BY

T. PRZYMUSIŃSKI (WARSZAWA)

In the present paper, all spaces are assumed to be Hausdorff. Notation and terminology are as in [4]. In particular, by a *mapping* we understand a continuous function and $f: X \twoheadrightarrow Y$ means that $f(X) = Y$.

The main purpose of the paper is to prove that an inverse image X of a locally connected metric space Y under open, perfect and 0-dimensional mapping $f: X \twoheadrightarrow Y$ is metrizable if $f^{-1}(y)$ is metrizable for every $y \in Y$ (in fact, the formulation will be a little more general). This is a solution of Problem 689 raised by Engelking and Lelek in [5].

Let us begin with a few historical remarks. In [1], Arkhangel'skiĭ showed that metrizability is an inverse invariant under open-and-closed finite-to-one mappings. In [7], Proizvolov proved that an inverse image of a metric space under a perfect 0-dimensional mapping is metrizable if it is locally connected.

Engelking and Lelek gave in [5] an example of a compact non-metrizable space X and of an open mapping $f: X \twoheadrightarrow Y$ onto the Cantor set Y with inverse images of points homeomorphic to the space $\{0, 1, \frac{1}{2}, \ldots\}$. From that example it follows that, in the Proizvolov theorem, the assumption of local connectedness of X cannot be omitted, even if the mapping is open and inverse images of points are countable.

Veličko in [8] found an example of a non-metrizable, compact, locally connected space X and of an open mapping $f: X \twoheadrightarrow I$ of X onto the unit interval $I = [0, 1]$, with inverse images of points homeomorphic to I. This example shows that in Proizvolov's theorem the condition that f is 0-dimensional is essential even if f is open.

Both these examples show that in Arkhangel'skiĭ's theorem the assumption that f is finite-to-one is important.

Now, we are going to prove that, by adding some other conditions in the Proizvolov theorem, we may assume that Y (instead of X) is locally connected.

First, we will give some definitions. We say that a mapping $f: X \twoheadrightarrow Y$

of a topological space X onto a topological space Y is *confluent* if for every connected closed subset C of Y, and any points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$, i. e., if every open-and-closed neighbourhood of x in $f^{-1}(C)$ meets $f^{-1}(y)$. This notion was introduced in [6]. We say that a mapping $f: X \twoheadrightarrow Y$ is *locally confluent* if every point $y \in Y$ has a neighbourhood V_y in Y such that $f|f^{-1}(V_y)$ is confluent. It is rather obvious that every open-and-closed mapping is confluent.

In [9], Zarelua defined the class of separative mappings. A mapping $f: X \twoheadrightarrow Y$ is called *separative* if for every point $x \in X$ and its neighbourhood U in X there exists a neighbourhood V of $f(x)$ in Y such that the set $f^{-1}(V)$ is not connected between $\{x\}$ and $f^{-1}(V) \setminus U$, i. e., that there is an open-and-closed neighbourhood of x in $f^{-1}(V)$ which is contained in U.

One can easily verify that a perfect mapping is separative if and only if it is 0-dimensional.

Now we formulate our theorem.

THEOREM 1. *Let $f: X \twoheadrightarrow Y$ be a separative locally confluent mapping of X onto a locally connected metrizable space Y. If for every y in a dense subset $A \subset Y$ the space $f^{-1}(y)$ is compact and metric, then X is metrizable.*

Proof. Zarelua noticed in [9] that if $f: X \twoheadrightarrow Y$ is separative and Y is regular, then X is also regular. We will define a σ-locally finite base in X.

Let $\mathfrak{V}$ be an open covering of Y such that $f|f^{-1}(V)$ is confluent for every $V \in \mathfrak{V}$. Since Y is locally connected, there exists, for every $n = 1, 2, \ldots$, an open covering $\mathfrak{G}_n = \{G_{n,s}\}_{s \in S_n}$ consisting of connected sets such that

(i) $\delta(G_{n,s}) < 1/n$ for every $s \in S_n$,

(ii) $\mathfrak{G}_n$ is a refinement of $\mathfrak{V}$.

Now, for every $n = 1, 2, \ldots$ and $s \in S_n$, we can find a $V_{n,s} \in \mathfrak{V}$ such that $G_{n,s} \subset V_{n,s}$; let us put $F_{n,s} = \overline{G_{n,s}} \cap V_{n,s}$. By the definition of $\mathfrak{V}$, the restriction $f|f^{-1}(F_{n,s})$ is confluent.

Let $\mathfrak{F}_{n,s}$ be the family of all non-empty open-and-closed subsets of $f^{-1}(F_{n,s})$. As in [5], Theorem 1, we shall prove that $\overline{\overline{\mathfrak{F}_{n,s}}} \leqslant \aleph_0$.

Let us take a $y_0 \in A \cap G_{n,s}$. The set $f^{-1}(y_0)$ is compact and metrizable, so the family of all its open-and-closed subsets is countable. Since $f|f^{-1}(F_{n,s})$ is confluent, every non-empty element of $\mathfrak{F}_{n,s}$ intersects $f^{-1}(y_0)$. Let B_1 and B_2 belong to $\mathfrak{F}_{n,s}$ and satisfy $B_1 \cap f^{-1}(y_0) = B_2 \cap f^{-1}(y_0)$. Then $B = B_1 \setminus B_2$ is open-and-closed in $f^{-1}(F_{n,s})$ and $B \cap f^{-1}(y_0) = 0$, which implies $B = 0$ and $B_1 = B_2$. This shows that $\overline{\overline{\mathfrak{F}_{n,s}}} \leqslant \aleph_0$. Let $\mathfrak{F}_{n,s} = \{Q_{n,s,m}\}_{m=1}^{\infty}$.

For $n = 1, 2, \ldots$, take an open locally finite refinement $\mathfrak{W}_n = \{W_{n,t}\}_{t \in T_n}$ of $\mathfrak{G}_n$ and for every $t \in T_n$ choose an $s(t) \in S_n$ such that $W_{n,s} \subset G_{n,s(t)} \subset F_{n,s(t)}$.

The family

$$\mathfrak{B}_{n,m} = \{f^{-1}(W_{n,t}) \cap Q_{n,s(t),m}\}_{t \in T_n}$$

is locally finite in X and consists of open sets.

If suffices to prove that $\mathfrak{B} = \bigcup_{n,m=1}^{\infty} \mathfrak{B}_{n,m}$ is a base of X. Since f is separative, for any point $x \in X$ and its neighbourhood U there exist a neighbourhood V of $f(x)$ in Y and an open-and-closed in $f^{-1}(V)$ set P such that $x \in P \subset U$. Since Y is regular, we may choose a neighbourhood V_0 of $f(x)$ satisfying $f(x) \in V_0 \subset \overline{V}_0 \subset V$. By virtue of (i), there exists an n_0 such that $\mathrm{St}(f(x), \mathfrak{G}_{n_0}) \subset V_0$. For any $W_{n_0,t_0} \in \mathfrak{W}_{n_0}$ containing $f(x)$, we have $f(x) \in W_{n_0,t_0} \subset G_{n_0,s(t_0)} \subset V_0$ and $G_{n_0,s(t_0)} \subset F_{n_0,s(t_0)} \subset \overline{G}_{n_0,s(t_0)} \subset \overline{V}_0 \subset V$.

Hence $f^{-1}(F_{n_0,s(t_0)}) \subset f^{-1}(V)$ and $x \in P \cap f^{-1}(F_{n_0,s(t_0)})$. Since the last set is open-and-closed in $f^{-1}(F_{n_0,s(t_0)})$, there exists an m_0 such that

$$Q_{n_0,s(t_0),m_0} = P \cap f^{-1}(F_{n_0,s(t_0)}).$$

We have then $x \in f^{-1}(W_{n_0,t_0}) \cap Q_{n_0,s(t_0),m_0} \subset P \subset U$ and $f^{-1}(W_{n_0,t_0}) \cap$ $\cap Q_{n_0,s(t_0),m_0} \in \mathfrak{B}_{n_0,m_0}$. This shows that $\mathfrak{B}$ is a base in X. The proof is completed.

Remarks made after Theorem 1 in [5] show that none of the assumptions in Theorem 1 can be omitted.

From Theorem 1 it follows

COROLLARY 1.1. *If there exists an open, perfect and 0-dimensional mapping $f: X \twoheadrightarrow Y$ of a space X onto a locally connected metrizable space Y, such that $f^{-1}(y)$ is metrizable for every $y \in Y$, then X is metrizable.*

Since every countable compact space is metrizable and 0-dimensional, we have

COROLLARY 1.2. *If there exists an open, perfect and countable-to-one mapping $f: X \twoheadrightarrow Y$ of a space X onto a locally connected metrizable space Y, then X is metrizable.*

Remark. One should compare Corollary 1.2 with the theorem of Arkhangel'skiĭ mentioned above. The example of Engelking and Lelek from [5] shows that the assumption of local connectedness of Y is essential.

The next two theorems show that the mapping f of Theorem 1 is necessarily open, and is closed if $A = Y$.

THEOREM 2. *Let $f: X \twoheadrightarrow Y$ be a separative, locally confluent mapping of X onto a locally connected regular space Y. If $f^{-1}(y_0)$ is compact for some $y_0 \in Y$, then f is closed at the point y_0.*

Proof. Let $f^{-1}(y_0) \subset U \subset X$, where U is open. We have to show that there exists an open set $G \subset Y$ such that $f^{-1}(y_0) \subset f^{-1}(G) \subset U$. Since f is separative, we may choose, for every $x \in f^{-1}(y_0)$, an open set

$V_x \subset Y$ and a $Q_x \subset f^{-1}(V_x)$ such that $y_0 \in V_x$, $x \in Q_x \subset U$ and Q_x is open-and-closed in $f^{-1}(V_x)$.

Let us choose a finite number of points $x_1, x_2, \ldots, x_n \in f^{-1}(y_0)$ such that $f^{-1}(y_0) \subset Q_{x_1} \cup Q_{x_2} \cup \ldots \cup Q_{x_n}$ and let W be a neighbourhood of the point y_0 in Y such that $f|f^{-1}(W)$ is confluent. By the regularity of Y there exists an open and connected set $V \subset Y$ satisfying $y_0 \in V \subset \overline{V} \subset \bigcap_{i=1}^{n} V_{x_i} \cap W$, whence $f^{-1}(y_0) \subset f^{-1}(\overline{V}) \subset f^{-1}(V_{x_i})$ for $i = 1, 2, \ldots, n$. The set $P_i = Q_{x_i} \cap f^{-1}(\overline{V})$ is open-and-closed in $f^{-1}(\overline{V})$ for $i = 1, 2, \ldots, n$, and $f^{-1}(y_0) \subset \bigcup_{i=1}^{n} P_i$. It follows then that $P = f^{-1}(\overline{V}) \setminus \bigcup_{i=1}^{n} P_i$ is open-and-closed in $f^{-1}(\overline{V})$ and that $P \cap f^{-1}(y_0) = 0$. Since $f|f^{-1}(\overline{V})$ is confluent, we have $P = 0$ and

$$f^{-1}(y_0) \subset f^{-1}(V) \subset f^{-1}(\overline{V}) = \bigcup_{i=1}^{n} P_i \subset \bigcup_{i=1}^{n} Q_{x_i} \subset U,$$

which completes the proof.

THEOREM 3. *Any separative, locally confluent mapping $f: X \twoheadrightarrow Y$ of X onto a locally connected, regular space Y is open.*

Proof. Let $\mathfrak{V}$ be an open covering of Y such that $f|f^{-1}(V)$ is confluent for every $V \in \mathfrak{V}$. Take an open base $\mathfrak{B} = \{B_s\}_{s \in S}$ of Y consisting of connected sets and such that for every $s \in S$ there exists a $V \in \mathfrak{V}$ satisfying $B_s \subset \overline{B}_s \subset V$. For every $s \in S$, let $\mathfrak{W}_s = \{W_{s,t}\}_{t \in T_s}$ be the family of all non-empty open-and-closed subsets of $f^{-1}(\overline{B}_s)$ and let $\mathfrak{U}_s = \{U_{s,t}\}_{t \in T_s}$, where $U_{s,t} = W_{s,t} \cap f^{-1}(B_s)$. Since f is separative, it is easy to see that $\mathfrak{U} = \bigcup_{s \in S} \mathfrak{U}_s$ is a base of X. And since $f|f^{-1}(\overline{B}_s)$ is confluent and $\overline{B}_s$ is connected for every $s \in S$, we infer that $f(W_{s,t}) = \overline{B}_s$ and $f(U_{s,t}) = f(W_{s,t}) \cap B_s = B_s$. Then, for every set U belonging to the base $\mathfrak{U}$ of X, the image $f(U)$ is open, i. e., f is open.

Theorems 2 and 3 show that Theorem 1 is "almost equivalent" to Corollary 1.1. More precisely, if in the formulation of Theorem 1 we assume that $A = Y$, then Theorem 1 coincides with Corollary 1.1.

Engelking and Lelek proved the following theorem ([5], Theorem 1):

A. *Let $f: X \twoheadrightarrow Y$ be a separative, locally confluent mapping of X onto a locally connected, regular space Y. If weight $w(Y) \geqslant \aleph_0$ and for every y in a dense subset $A \subset Y$ the inverse image $f^{-1}(y)$ is compact and $w(f^{-1}(y)) \leqslant w(Y)$, then $w(X) \leqslant w(Y)$.*

Here the situation is similar: if in this theorem we assume that $A = Y$, then it is equivalent to the following particular case ([5], Corollary 1.2):

B. *Let $f: X \twoheadrightarrow Y$ be an open, perfect and 0-dimensional mapping of X onto a regular locally connected space Y. If $w(Y) \geqslant \aleph_0$ and $w(f^{-1}(y)) \leqslant w(Y)$ for every $y \in Y$, then $w(X) \leqslant w(Y)$.*

We shall show that in B the space Y may be assumed to fulfill a condition somewhat weaker than local connectedness. Let us begin with the definition.

Definition. We call a topological space X *weakly locally connected* (w. l. c.) if it admits a grid $\mathfrak{N}$ of power not greater than $w(X)$, consisting of connected sets.

By a *grid* in X we mean a family $\mathfrak{N}$ of subsets of X such that for every $x \in X$ and its neighbourhood U there exists $N \in \mathfrak{N}$ with $x \in N \subset U$.

Remarks. 1. X is *w. l. c.* if and only if every open set in X has at most $w(X)$ components.

2. If X is locally connected, then X is w. l. c.

3. If $w(X) \geqslant \overline{\overline{X}}$, then X is w. l. c., for example "the double segment of Alexandroff" (see [4], Example 3.1.2).

4. The space $X = R \setminus \{1/n : n = 1, 2, \ldots.\}$, where R denotes the real line, is w. l. c., although $w(X) < \overline{\overline{X}}$ and X is not locally connected.

THEOREM 4. *Let $f: X \twoheadrightarrow Y$ be an open, perfect and 0-dimensional mapping of X onto a weakly locally connected space Y. If $w(Y) \geqslant \aleph_0$ and $w(f^{-1}(y)) \leqslant w(Y)$ for every $y \in Y$, then $w(X) \leqslant w(Y)$.*

Proof. A lemma proved by Arkhangel'skiĭ ([3], Lemma 4) shows that it is enough to prove that X admits a grid of power not greater than $w(Y)$.

Let $\mathfrak{N} = \{N_s\}_{s \in S}$ be a grid in Y consisting of connected sets such that $\overline{\overline{S}} \leqslant w(Y)$. For every $s \in S$ let $\mathfrak{C}_s = \{C_{s,t}\}_{t \in T_s}$ be the family of all non-empty open-and-closed sets in $f^{-1}(N_s)$. As in Theorem 1, we prove that $\overline{\overline{T}}_s \leqslant w(Y)$ for every $s \in S$ and that $\mathfrak{C} = \bigcup_{s \in S} \mathfrak{C}_s$ is a grid in X. The theorem follows from the obvious inequality $\overline{\overline{\mathfrak{C}}} \leqslant w(Y)$.

COROLLARY 4.1. *Let $f: X \twoheadrightarrow Y$ be an open, perfect and countable-to-one mapping of X onto a weakly locally connected space Y. Then $w(X) \leqslant w(Y)$, if $w(Y) \geqslant \aleph_0$.*

Remarks. 1. From [5] and well-known examples of open finite-to-one mappings of countable, non-metrizable $T_{3\frac{1}{2}}$-spaces onto countable metrizable spaces it follows that none of the assumptions in Theorem 4 can be omitted. One should compare Corollary 1.4 with the theorem of Arkhangel'skiĭ proved in [3], which says that infinite weight of a topological space is an inverse invariant under open-and-closed, finite-to-one mappings. The example from [5] shows that the assumption that Y is w. l. c. is essential.

2. Not every w. l. c. metrizable space has a σ-discrete grid consisting of connected sets (for example, the union $\bigoplus_{r \in R} C_r$ of continuum copies of the Cantor set), but obviously every w. l. c. separable metric space has. From [2] (Theorem 5.3, p. 167, and Theorem 5.2, p. 165) we infer that

if $f: X \to Y$ is an open, perfect and 0-dimensional mapping of X onto a metric space Y which admits a σ-discrete grid consisting of connected sets, and if $f^{-1}(y)$ is metrizable for every $y \in Y$, then X is metrizable. Combining the example from [5] with the above example of w. l. c. metric space without any σ-discrete grid consisting of connected sets, we can see that if we assume only that Y is metric and w. l. c., the theorem becomes false. Hence the counterpart of Theorem 4 for metrizability does not hold.

3. In a similar way, we infer that Theorems 2 and 3 become false if we assume only that Y is w. l. c. and regular.

The author is grateful to Prof. R. Engelking for his help in the preparation of this paper.

REFERENCES

[1] А. В. Архангельский, *Теорема о метризуемости прообраза метрического пространства при открыто-замкнутом конечно-кратном отображении, пример и нерешенные задачи*, Доклады Академии Наук СССР 170 (1966), p. 754-762.

[2] — *Отображения и пространства*, Успехи Математических Наук 21 (1966), выпуск 4, p. 133-184.

[3] — *Признак существования бикомпактного элемента в непрерывном разбиении. Теорема об инвариантности веса при открыто-замкнутом конечно-кратном отображении*, Доклады Академии Наук СССР 166 (1966), p. 1263-1266.

[4] R. Engelking, *Outline of general topology*, Amsterdam 1968.

[5] — and A. Lelek, *Metrizability and weight of inverses under confluent mapping*, Colloquium Mathematicum 21 (1970), p. 239-246.

[6] A. Lelek, *On confluent mappings*, ibidem 15 (1966), p. 223-233.

[7] В. В. Произволов, *О разбивающих отображениях локально связных пространств*, Доклады Академии Наук СССР 159 (1964), p. 516-518.

[8] Н. В. Величко, *Пример открытого компактного монотонного непрерывного отображения неметризуемого бикомпакта на компакт*, ibidem 177 (1967), p. 995-996.

[9] А. Зарелуа, *Равенство размерностей и бикомпактные расширения*, ibidem 144 (1962), p. 713-716.

WARSAW UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS

Reçu par la Rédaction le 7. 12. 1970