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ON ALMOST-PERIODIC SEQUENCES

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Let X denote a metric space with the distance $|x-y|$ and $b(X)$ the space of sequences $\xi = \{x_1, x_2, \ldots\}$ bounded in the space X (with the metric $|\xi-\eta| = \sup\limits_k |x_k - y_k|$).

Definition. A sequence $\xi = \{x_k\} \epsilon b(X)$ is said to be *almost-periodic* if the set of sequences $\xi_p = \{x_p, x_{p+1}, \ldots\}$, $p = 1, 2, \ldots$, is precompact in the space $b(X)$.

Remark. A sequence so defined is an almost-periodic function on the semigroup of positive integers. It is well known and easy to prove that for every such sequence $\{x_1, x_2, \ldots\}$ there exists a sequence $\{\ldots, y_{-1}, y_0, y_1, \ldots\}$, almost-periodic as a function on the group of integers, such that $x_k - y_k \to 0$ as $k \to \infty$.

It is evident that 1° any almost-periodic sequence is also precompact in X, i. e. every its subsequence contains a fundamental subsequence, and that 2° any convergent sequence is almost-periodic.

LEMMA. *A sequence $\xi = \{x_k\}$ is almost-periodic if and only if*

(Ω) *for any $\varepsilon > 0$ there exist a sequence of integers $\{N_k^{(\varepsilon)}\}$ and two constants K_ε and M_ε such that $0 < N_{k+1}^{(\varepsilon)} - N_k^{(\varepsilon)} < M_\varepsilon$ $(k = 1, 2, \ldots)$ and*

$$|x_{n+N_k^{(\varepsilon)}} - x_n| < \varepsilon \quad \textit{for} \quad n > K_\varepsilon,\ j = 1, 2, \ldots$$

The proof of this lemma is analogous to the proof of a well-known theorem of Bochner [1].

The purpose of this paper is to prove the following

THEOREM. *For any sequence $\{x_k\}$, precompact in X, there exists a permutation $\varphi(k)$ of the set of positive integers such that the sequence $y_k = x_{\varphi(k)}$ is almost-periodic.*

Proof. Let the sequence $\xi = \{x_k\}$ be precompact in X. By E we denote the *graph* of ξ (this means that terms of ξ with distinct indices

[1] See for example N. Dunford and T. Schwartz, *Linear operators*, New York-London 1958, § IV, 7, Theorem 2.

are considered as distinct elements of the set E). Let Z_n be the set of those terms x of the sequence ξ for which the sphere with centre x and radius 2^{-n} contains only a finite number of elements of ξ. It is easily seen that all the sets Z_n are finite.

A finite system $A = \{x_1, x_2, \ldots, x_p\}$ shall be called an *ε-multiple* of a system $B = \{y_1, y_2, \ldots, y_r\}$ if

1° $p = nr$ (n is an integer $\geqslant 1$),

2° $|y_i - x_{i+\nu r}| < \varepsilon$ for $i = 1, 2, \ldots, r$; $\nu = 1, 2, \ldots, n-1$.

We shall construct the required sequence $\{y_k\} = \{x_{\varphi(k)}\}$ step by step. A set A is termed to be an *ε-network* for the set B, when $A \subset B$ and for every $b \in B$ there exists an $a \in A$ such that $|a-b| < \varepsilon$.

Let now $U_1 = Z_1 = \{y_1, \ldots, y_{m_1}\}$ and $V_1 = \{y_{m_1+1}, \ldots, y_{n_1}\}$ be a $\frac{1}{2}$-network for the set $E \setminus U_1$ such that $x_1 \in U_1 \cup V_1$. Let $U_2 = \{y_{n_1+1}, \ldots, y_{m_2}\}$ be a $\frac{1}{2}$-multiple of the set V_1 and simultaneously a $\frac{1}{2}$-network for the set $E \setminus (U_1 \cup V_1)$ such that

$$Z_2 \setminus (U_1 \cup V_1) \subset U_2 \subset E \setminus (U_1 \cup V_1).$$

We shall prove that such a set exists. In fact, we may choose a finite $\frac{1}{2}$-network for the set $E \setminus (U_1 \cup V_1)$ which contains all points of the set $Z_2 \setminus (U_1 \cup V_1)$. Next we may extend this network to a $\frac{1}{2}$-multiple of the set V_1. This is possible because the set V_1 is a $\frac{1}{2}$-network of $E \setminus (U_1 \cup V_1)$ and it is disjoint with the set Z_1.

Let further $V_2 = \{y_{m_2+1}, \ldots, y_{n_2}\}$ be a $\frac{1}{4}$-network of the set $E \setminus (U_1 \cup V_1 \cup U_2)$ and simultaneously a $\frac{1}{2}$-multiple of the set U_2 and such that $V_2 \cap Z_2 = \emptyset$, $x_2 \in U_1 \cup V_1 \cup U_2 \cup V_2$. The set V_2 may be constructed in the following way. We choose a finite $\frac{1}{4}$-network of the set $E \setminus (U_1 \cup V_1 \cup U_2)$ which may or may not contain the point x_2 and next extend this network to a $\frac{1}{2}$-multiple of the set U_2. This is possible because the set U_2 is disjoint with the set Z_1 and it is a $\frac{1}{2}$-network for $E \setminus (U_1 \cup V_1)$. Suppose the systems $U_1, V_1, \ldots, U_s, V_s$ have been defined. We define the system $U_{s+1} = \{y_{n_s+1}, \ldots, y_{m_{s+1}}\}$ as the 2^{-s} network for the set $E \setminus (U_1 \cup \ldots \cup U_s \cup V_1 \cup \ldots \cup V_s)$ being simultaneously a 2^{-s} multiple of the system V_s such that

$$Z_{s+1} \setminus \bigcup_{i=1}^{s} (U_i \cup V_i) \subset U_{s+1}.$$

We define the system $V_{s+1} = \{y_{m_{s+1}+1}, \ldots, y_{n_{s+1}}\}$ as a 2^{-s-1}-network of the set $E \setminus (U_1 \cup \ldots \cup U_{s+1} \cup V_1 \cup \ldots \cup V_s)$ which is simultaneously a 2^{-s} multiple of the system U_{s+1} such that

$$x_{s+1} \in \bigcup_{i=1}^{s+1} (U_i \cup V_i).$$

The inductively defined sequence of systems $U_1, V_1, U_2, V_2, \ldots$ determines the sequence $\{y_k\} = \{x_{\varphi(k)}\}$.

We shall prove that the sequence $\{y_k\}$ satisfies the condition (Ω) of the lemma. To this end let K_n, for $n = 1, 2, \ldots$, be the n-th term of the sequence of systems $V_\mu, U_{\mu+1}, V_{\mu+1}, U_{\mu+2}, \ldots$

Next let $|K_n|$ denote the length of the system K_n. We shall define a function $q = \psi(p)$ for $p = 2, 3, \ldots$ Let y_l be a fixed element of the system V_μ and let a denote the length of the system V_μ, i. e. let $a = |K_\mu| = n_\mu - m_\mu$. If $y_{l+pa} \in K_r$, then $\psi(p) = q$ is defined to be a non-negative integer for which

$$(*) \qquad y_{l+qa} \in K_{r-1},$$

and

(**) there exists a positive integer s such that $qa + s|K_{r-1}| = pa$.

It is obvious that the function ψ is well defined. Let us remark that for every p there exists a ν such that $\psi^\nu(p) = 0$, where ψ^ν denotes the ν-th iteration of ψ. If K_r is a $2^{-\beta}$-multiple of K_{r-1}, then

$$|y_{l+pa} - y_{l+\psi(p)a}| < 2^{-\beta}.$$

Hence we have

$$|y_{l+pa} - y_l| < 2\sum_{j=\mu}^{\infty} 2^{-j}$$

for every p.

Now, let $n > n_\mu$. Then there exists a positive integer p such that $y_n = y_{l+pa}$, where $x_l \in V_\mu$. Therefore we have

$$(1) \qquad |y_n - y_{n+ka}| \leqslant |y_{l+pa} - y_l| + |y_{l+(p+k)a} - y_l| < 4\sum_{l=\mu}^{\infty} 2^{-j}.$$

Let now $\varepsilon > 0$ and μ be such that

$$4\sum_{j=\mu}^{\infty} 2^{-j} < \varepsilon.$$

In view of (1) condition (Ω) is satisfied with

$$K_\varepsilon = n_\mu, \qquad M_\varepsilon = n_\mu - m_\mu, \qquad N_k^{(\varepsilon)} = k(n_\mu - m_\mu), \qquad k = 1, 2, \ldots$$

This completes the proof of the theorem.

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