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VON NEUMANN ALGEBRAS GENERATED BY REPRESENTATIONS OF NILPOTENT GROUPS

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In this paper we prove the following

THEOREM. *Let $\{U_g\}''_{g\in G}$ be the von Neumann algebra generated by operators of a unitary representation $g \to U_g$ of a countable nilpotent group G in a separable Hilbert space. If $\{U_g\}''_{g\in G}$ has a finite, faithful, normal trace, then it is hyperfinite.*

This theorem was announced in [4] but no complete proof has been published. Here we present such a proof which makes use of the theory of Dye [2], [3]. The theorem is also a consequence of a general theory of A. Connes published in Annals of Mathematics 104 (1976), p. 73-115. The proof we offer seems to be much simpler and more direct than the one using Connes' theory.

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Professor M. Takesaki called our attention to the fact that the assumption that $\{U_g\}''$ is finite is redundant, since under the assumption that $\{U_g\}''$ is infinite the proof of the theorem, as announced in [5], has been completed in [1].

Let $\mathfrak{A}$ be a von Neumann algebra of linear bounded operators in a separable Hilbert space and let tr be a finite, faithful, normal trace on $\mathfrak{A}$. The equality $(a, b) = \mathrm{tr}\, b^*a$ defines an inner product in $\mathfrak{A}$. The completion $H(\mathfrak{A})$ of $\mathfrak{A}$ with respect to the trace norm $\|a\| = (a, a)^{1/2}$ is a Hilbert space. The mapping $a \to l_a$, where $l_a x = ax$, is an isomorphism of $\mathfrak{A}$ into the algebra $B(H(\mathfrak{A}))$ of all linear bounded operators in $H(\mathfrak{A})$. We denote by $\mathrm{E}_{\mathfrak{B}}$ the conditional expectation of $\mathfrak{A}$ with respect to a von Neumann subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (cf. [6]). Linearity and the following properties of $\mathrm{E}_{\mathfrak{B}}$ will be used:

$$\mathrm{E}_{\mathfrak{B}}(a^*) = (\mathrm{E}_{\mathfrak{B}}(a))^*, \quad \mathrm{E}_{\mathfrak{B}}(b_1ab_2) = b_1\mathrm{E}_{\mathfrak{B}}(a)b_2, \quad \mathrm{tr}\, a = \mathrm{tr}\,\mathrm{E}_{\mathfrak{B}}(a) \quad \text{for } a \in \mathfrak{A},\ b_1, b_2 \in \mathfrak{B}.$$

An algebra $\mathfrak{A}$ is called *hyperfinite* if for given $a_i \in \mathfrak{A}$, $1 \leqslant i \leqslant k$, and $\varepsilon > 0$ there is a subalgebra $\mathfrak{N}$ of type I of $\mathfrak{A}$ which contains the center $\mathfrak{Z}(\mathfrak{A})$ of $\mathfrak{A}$ and elements $n_i \in \mathfrak{N}$ such that

$$\|a_i - n_i\| < \varepsilon \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $\operatorname{aut}\mathfrak{Z}(\mathfrak{A})$ be the group of all trace and $*$-preserving automorphisms of $\mathfrak{Z}(\mathfrak{A})$. For $\alpha, \beta \in \operatorname{aut}\mathfrak{Z}(\mathfrak{A})$ we denote by $F(\alpha, \beta)$ the greatest projection P in $\mathfrak{Z}(\mathfrak{A})$ such that $Q^\alpha = Q^\beta$ for all projections $Q \leqslant P$. In the case $F(\alpha, 1) = 0$ we say that α is *freely acting*. Let S be a subgroup of $\operatorname{aut}\mathfrak{Z}(\mathfrak{A})$. We say that an automorphism α *belongs to* $[S]$ if

$$\operatorname*{LUB}_{s \in S} F(\alpha, s) = 1.$$

The following (i)-(iv) give a description of $[S]$ which is due to Dye [2], [3].

(i) $\alpha \in [S]$ if and only if there is a partition $\{P(s, \alpha)\}_{s \in S}$ of 1 in $\mathfrak{Z}(\mathfrak{A}) = \mathfrak{Z}$ (i.e. $P(s, \alpha)$ are mutually orthogonal projections and $\sum_s P(s, \alpha) = 1$) such that $P(s, \alpha) \leqslant F(s, \alpha)$ for $s \in S$ and

$$P^\alpha = \sum_s \big(P(s, \alpha)P\big)^s \quad \text{for all projections } P \in \mathfrak{Z}.$$

(ii) If S is an Abelian subgroup of $\operatorname{aut}\mathfrak{Z}$, then there exists a partition $\{D_n\}_{n=0}^\infty$ of 1 in $\mathfrak{Z}$ such that $D_n^s = D_n$ for all $n \geqslant 0$, $s \in S$ and

(a) $[S]$ is a direct product of groups $[S]_{D_n}$, $n = 0, 1, 2, \ldots$, where

$$[S]_{D_n} = \{\alpha \in [S] : 1 - D_n \leqslant F(\alpha, 1)\};$$

(b) $[S]_{D_n} = [Z_n]$ for all $n = 1, 2, 3, \ldots$, where Z_n is the cyclic group of order n of freely acting automorphisms of the algebra $\mathfrak{Z}D_n$;

(c) $[S]_{D_0} = [Z_2^{\aleph_0}]$, where $Z_2^{\aleph_0}$ is the restricted direct product of $\aleph_0$ copies of Z_2 of freely acting automorphisms of $\mathfrak{Z}D_0$.

(iii) If S is a finite group of freely acting automorphisms of $\mathfrak{Z}$, then there is a projection P in $\mathfrak{Z}$ such that $\{P^s\}_{s \in S}$ is a partition of 1.

(iv) If S is a finite group of freely acting automorphisms of $\mathfrak{Z}$, then there exists a cyclic group Z_n of freely acting automorphisms of $\mathfrak{Z}$ such that $[S] = [Z_n]$.

We have also

(v) Let s be a freely acting automorphism of $\mathfrak{Z}(\mathfrak{A})$. If for an element $a \in \mathfrak{A}$ the equality $a(z^s - z) = 0$ holds for all $z \in \mathfrak{Z}(\mathfrak{A})$, then $a = 0$.

In fact, since the equalities $P(P^s - P) = 0$ and $\big(P(P - P^{s^{-1}})\big)^s = 0$ hold for $P \leqslant c(a)$ ($c(a)$ is the central support of a), we have $P = P^s$. Hence $c(a) \leqslant F(s, 1) = 0$ and, consequently, $a = 0$.

From now on, all von Neumann algebras which will be dealt with have finite, faithful, normal trace.

LEMMA 1. *Let* $\mathfrak{B} = \{\mathfrak{A}, \{U_s\}_{s\in S}\}''$ *be the von Neumann algebra generated by a von Neumann algebra* $\mathfrak{A}$ *and a family* $\{U_s\}_{s\in S}$ *of unitaries, where* S *is a subgroup of* $\operatorname{aut}\mathfrak{Z}(\mathfrak{A})$, $U_1 = 1$, $U_s^*\mathfrak{A}U_s = \mathfrak{A}$, $U_s^*zU_s = z^s$, $U_{st}^*U_sU_t = a(s,t) \in \mathfrak{A}$ *for all* $z \in \mathfrak{Z}(\mathfrak{A})$, $s, t \in S$. *Let* $[S] = [K]$, *where* K *is a group of freely acting automorphisms of* $\mathfrak{Z}(\mathfrak{A})$. *Then there is a family of unitaries* $\{U_k\}_{k\in K}$ *in* $\mathfrak{B}$ *such that each* $b \in \mathfrak{B}$ *has a representation of the form*

$$b = \sum_{k\in K} a_k U_k,$$

where $a_k \in \mathfrak{A}$ *and the summation being taken in the trace norm* $\|\cdot\|$. *Further, we have* $\mathfrak{Z}(\mathfrak{A})' \cap \mathfrak{B} = \mathfrak{A}$.

Proof. Let $a \in [S]$. By (i),

$$P^a = \sum_{s\in S} (P(s,a)P)^s,$$

where $P \in \mathfrak{Z}(\mathfrak{A})$, $\{P(s,a)\}_{s\in S}$ is a partition of 1. Let us put

$$U_a = \sum_{s\in S} P(s,a)U_s. \tag{1}$$

We have

$$\begin{aligned} U_aU_a^* &= \sum_{s,t\in S} P(s,a)U_sU_t^*P(t,a) = \sum_{s,t} U_sP(s,a)^sP(t,a)^tU_t^* \\ &= \sum_{s,t} U_s(P(s,a)P(t,a))^aU_t^* = \sum_s U_sP(s,a)^aU_s^* = \sum_s P(s,a) = 1, \end{aligned}$$

since $P(s,a) \leqslant F(s,a)$. Similarly, if $a \in \mathfrak{A}$, then

$$\begin{aligned} U_a^*aU_a &= \sum_{s,t\in S} U_s^*P(s,a)aP(t,a)U_t = \sum_{s,t} U_s^*P(s,a)P(t,a)aU_t \\ &= \sum_s U_s^*P(s,a)aU_s. \end{aligned}$$

Consequently, $U_a^*\mathfrak{A}U_a = \mathfrak{A}$ and, putting $a = 1$, we have $U_aU_a^* = U_a^*U_a = 1$, i.e. the operator U is unitary. Moreover, $U_a^*PU_a = P^a$ for all projections $P \in \mathfrak{Z}(\mathfrak{A})$.

For $t \in S$ and $k \in K$ let $\{P_t(s,k)\}_{s\in S}$ be a family of projections defined by the following condition:

$\{P_t(s,k)\}_{s\in S}$ is a maximal family of mutually orthogonal projections such that $P_t(t,k) = F(t,k)$ and $P_t(s,k) \leqslant F(s,k)$ for $s \in S$.

Since $k \in [S]$, it is clear that

$$\sum_{s\in S} P_t(s,k) = 1.$$

Now for the partition $\{P_t(s, k)\}_{s\in S}$ let $U_k(t)$ be defined by (1). Since $t \in [K]$, we have

$$(2) \quad \sum_{k\in K} P_t(t, k)\, U_k(t) = \sum_{\substack{k\in K\\ s\in S}} P_t(t, k) P_t(s, k)\, U_s = \sum_{k\in K} F(t, k)\, U_t = U_t$$

and, consequently, for $t_1, t_2 \in S$ and $k \in K$ we have

$$\begin{aligned} U_k(t_1)\, U_k(t_2)^* &= \sum_{s,t\in S} P_{t_1}(s, k)\, U_s U_t^* P_{t_2}(t, k) \\ &= \sum_{s,t} P_{t_1}(s, k) P_{t_2}(t, k)^{ts^{-1}} U_s U_t^* \\ &= \sum_{s,t} P_{t_1}(s, k)^{ks^{-1}} P_{t_2}(t, k)^{ks^{-1}} U_{st^{-1}} a(s, t^{-1}) a(t, t^{-1})^* \\ &= \sum_{l\in K} \sum_{s,t\in S} P_{t_1}(s, k) P_{t_2}(t, k) P_{st^{-1}}(st^{-1}, l)\, U_l a(s, t^{-1}) a(t, t^{-1})^*. \end{aligned}$$

Since

$$\begin{aligned} P_{t_1}(s, k) P_{t_2}(t, k) P_{st^{-1}}(st^{-1}, l) &\leqslant F(s, k) F(t, k) F(st^{-1}, l) \\ &\leqslant F(st^{-1}, 1) F(st^{-1}, l) \leqslant F(l, 1) \end{aligned}$$

and $F(l, 1) = 0$ for each l from K such that $l \neq 1$, we see that $U_k(t_1)\, U_k(t_2)^*$ is an $\mathfrak{A}$.

Let us put $U_k = U_k(1)$. Then we obtain a family $\{U_k\}_{k\in K}$ of unitaries in $\mathfrak{B}$. Let D be the set of finite sums of the form $\sum_{k\in K} a_k U_k$, where $a_k \in \mathfrak{A}$. We show that D is an involutive algebra. Indeed, we have

$$\begin{aligned} U_{kl}^* U_k U_l &= \sum_{s,t,w\in S} U_s^* P_1(s, kl) P_1(t, k)\, U_t P_1(w, l)\, U_l \\ &= \sum_{s,t,w} P_1(s, kl)^s P_1(t, k)^s P_1(w, l)^{t^{-1}s} U_s^* U_t U_l \\ &= \sum_{s,t,w} P_1(s, kl)^s P_1(t, k)^s P_1(w, l)^{t^{-1}s} U_{s^{-1}tl} a'(s, t, l) \\ &= \sum_{\substack{s,t,w\in S\\ m\in K}} P_1(s, kl)^s P_1(t, k)^s P_1(w, l)^{t^{-1}s} P_{s^{-1}tl}(s^{-1}tl, m)\, U_m a'(s, t, l), \end{aligned}$$

where $a'(s, t, l)$ are unitaries from $\mathfrak{A}$. Let Q be a projection from $\mathfrak{Z}(\mathfrak{A})$ such that

$$Q \leqslant P_1(s, kl)^s P_1(t, k)^s P_1(w, l)^{t^{-1}s} P_{s^{-1}tl}(s^{-1}tl, m).$$

Then we have $Q^{s^{-1}kls^{-1}} = Q^{s^{-1}}$, $Q^{s^{-1}lk^{-1}} = Q^{s^{-1}}$, $Q^{s^{-1}ll} = Q^{m}$, which shows that $Q^m = Q$ and, therefore, $Q \leqslant F(m, 1)$. Since m $(m \neq 1)$ is freely acting, we see that $U_{kl}^* U_k U_l$ is in $\mathfrak{A}$ and, consequently, D is an involutive subalgebra of $\mathfrak{B}$. Moreover, it follows from (2) that $U_s \in D$ for all $s \in S$ and, therefore, $D'' = \mathfrak{B}$.

Our justification for above calculating is that all series, which appeared, converge in the operator norm.

Now, let us suppose that $bz = zb$ for some $b \in \mathfrak{B}$ and $z \in \mathfrak{Z}(\mathfrak{A})$. Applying conditional expectation $\mathrm{E}_{\mathfrak{A}}$ to the identity

$$bU_k^* U_l U_m^* z - z^{ml^{-1}k} bU_k^* U_l U_m^* = 0,$$

we get

$$\mathrm{E}_{\mathfrak{A}}(bU_k^* U_l U_m^*)(z^{ml^{-1}k} - z) = 0, \tag{3}$$

provided $bz = zb$, $b \in \mathfrak{B}$, $z \in \mathfrak{Z}(\mathfrak{A})$, $k, l, m \in K$.

If we put $b = (a_l^* a_k)^l$ and $U_m = 1$, then we obtain

$$0 = (a_l^* a_k)^l\, \mathrm{E}_{\mathfrak{A}}(U_l^* U_k) = \mathrm{tr}\, \mathrm{E}_{\mathfrak{A}}(U_l^* a_l^* a_k U_k) = (a_k U_k, a_l U_l)$$

in view of (v). Thus $\{\mathfrak{A} U_k\}_{k \in K}$ is a mutually orthogonal family of subspaces of $H(\mathfrak{B})$. Therefore, the $\|\cdot\|$-closures $H(\mathfrak{A}) U_k$ of $\mathfrak{A} U_k$ form a family of mutually orthogonal closed subspaces of $H(\mathfrak{B})$. Of course, D is $\|\cdot\|$-dense in $H(\mathfrak{B})$, since $D'' = \mathfrak{B}$ and $d \to b$ (strong) implies $\|b - d\| \to 0$. Therefore, $H(\mathfrak{B})$ is the orthogonal sum of subspaces $\{H(\mathfrak{B}) U_k\}_{k \in K}$.

Consider the mapping $\mu_k \colon \mathfrak{B} \to H(\mathfrak{A}) U_k$ defined by

$$\mu_k(b) = \mathrm{E}_{\mathfrak{A}}(bU_k^*) U_k, \quad \text{where} \quad k \in K.$$

If we put $b = a_l$ and $U_k = 1$ into (3), then we get

$$\mathrm{E}_{\mathfrak{A}}(a_l U_l U_m^*) = \delta_{lm^{-1}} a_l$$

because of (v). Thus

$$\mu_m(a_l U_l) = \mathrm{E}_{\mathfrak{A}}(a_l U_l U_m^*) U_m = \delta_{lm^{-1}} a_l U_l,$$

which shows that the restriction of μ_m to the elements from D is the orthogonal projection into $H(\mathfrak{A}) U_m$. Since the mapping μ_m is $\|\cdot\|$-continuous and D is $\|\cdot\|$-dense in $H(\mathfrak{B})$, $\mu_m(b)$ is the projection of an element $b \in \mathfrak{B}$ on the space $H(\mathfrak{A}) U_m$ and, consequently, there are elements a_k is $\mathfrak{A}$, namely $a_k = \mathrm{E}_{\mathfrak{A}}(bU_k^*)$, such that

$$b = \sum_k a_k U_k.$$

If we put $U_l = U_m = 1$ into (3), then, taking into account (v), we get $\mathrm{E}_{\mathfrak{A}}(bU_k^*) = 0$ for all $k \in K$, $k \neq 1$, provided $bz = zb$ for all $z \in \mathfrak{Z}(\mathfrak{A})$. Consequently, $b = \mathrm{E}_{\mathfrak{A}}(b)$ which belongs to $\mathfrak{A}$ and Lemma 1 follows.

LEMMA 2. *Let* $\mathfrak{B} = \{\mathfrak{A}, \{U_s\}_{s\in S}\}''$, *where* $U_s^*\mathfrak{A}U_s = \mathfrak{A}$, $U_s^*zU_s = z^s$, $S < \operatorname{aut}\mathfrak{Z}(\mathfrak{A})$ *and suppose that the unitaries* U_s *satisfy the conditions* $U_1 = 1$ *and* $U_{st}^*U_sU_t \in \mathfrak{A}$ *for* $s, t \in S$, $z \in \mathfrak{Z}(\mathfrak{A})$. *Let* $[S] = [K]$, *where* K *is a finite cyclic group of freely acting automorphisms of* $\mathfrak{Z}(\mathfrak{A})$. *Then, if* $\mathfrak{A}$ *is hyperfinite, so is* $\mathfrak{B}$.

Proof. By Lemma 1, every $b \in \mathfrak{B}$ has a representation of the form $\sum_k a_kU_k$. Let

$$\varepsilon > 0 \quad \text{and} \quad b_p = \sum_k a'_{k,p}U_k, \quad 0 \leqslant p \leqslant n,$$

be given. In view of (iii), there exists a projection P in $\mathfrak{Z}(\mathfrak{A})$ such that $\{P^k\}_{k\in K}$ is a partition of 1. Let g be a generator of the cyclic group K of order m. It follows from Lemma 1 that $U_{g^i}U_g^i \in \mathfrak{A}$, $i = 0, 1, \ldots, m-1$, and, therefore, $U_g^m \in \mathfrak{A}$. Let us put

$$U = \exp\left[\frac{-iA}{m}\right],$$

where A is self-adjoint such that $U_g^m = \exp[iA]$. Of course, $U \in \mathfrak{A}$. We have also $UU_g = U_gU$ and, consequently, $(U_gU)^m = U_g^mU^m = 1$. It follows from Lemma 1 that

$$b_p = \sum_k a_{k,p}V_k,$$

where $V_k = (U_gU)^i$, $k = g^i$, $i = 0, 1, \ldots, m-1$, and $a_{k,p} \in \mathfrak{A}$.

Since $\mathfrak{A}$ is hyperfinite, so is $\mathfrak{A}P$. Therefore, for $\varepsilon_1 > 0$ and the elements $a_{k,p}^lP \in \mathfrak{A}P$, where $k, l \in K$ and $0 \leqslant p \leqslant n$, there is a subalgebra $\mathfrak{N}_1$ of type I of $\mathfrak{A}P$, which contains the center of $\mathfrak{A}P$ and elements $n_{k,l,p}$ in $\mathfrak{N}_1$ such that

$$\|a_{k,p}^lP - n_{k,l,p}\| < \varepsilon_1 \quad \text{for } k, l \in K,\ 0 \leqslant p \leqslant n.$$

Let us put $\mathfrak{N} = \{\mathfrak{N}_1, V_g\}''$. The algebra $\mathfrak{N}$ is of type I and $\mathfrak{Z}(\mathfrak{B}) \subset \mathfrak{N}$. In fact, by Lemma 1, the center of $\mathfrak{B}$ consists of elements $z \in \mathfrak{Z}(\mathfrak{A})$ such that $z^k = z$ for all $k \in K$. Since

$$zP \in \mathfrak{N}_1 \quad \text{and} \quad z = \sum_k (zP)^k,$$

we see that z is in $\mathfrak{N}$. Now, for the elements

$$\sum_{k,l\in K} n_{k,l,p}^{l^{-1}}V_k \in \mathfrak{N}$$

we have

$$\Big\|b_p - \sum_{k,l} n_{k,l,p}^{l-1} V_k\Big\| = \Big\| \sum_k a_{k,p} V_k - \sum_{k,l} n_{k,l,p}^{l-1} V_k\Big\|$$

$$\leqslant \sum_k \|(a_{k,p}^l P)^{l-1} - n_{k,l,p}^{l-1}\| \leqslant \sum_{k,l} \|a_{k,p}^l P - n_{k,l,p}\| \leqslant \sum_{k,l} \varepsilon_1,$$

which completes the proof of Lemma 2.

LEMMA 3. *Let* $\mathfrak{B} = \{\mathfrak{A}, \{U_s\}_{s\in S}\}''$, *where* $U_s^*\mathfrak{A}U_s = \mathfrak{A}$, $U_s z U_s = z^s$, $z \in \mathfrak{Z}(\mathfrak{A})$, $s \in S$, *S being a countable Abelian subgroup of* $\operatorname{aut}\mathfrak{Z}(\mathfrak{A})$. *Let us suppose that the unitaries U_s satisfy $U_1 = 1$ and $U_{st}^* U_s U_t \in \mathfrak{A}$ for $s, t \in S$. Then, if $\mathfrak{A}$ is hyperfinite, so is $\mathfrak{B}$.*

Proof. Let $\{D_n\}_{n=0}^{\infty}$ be the partition of 1 as in (ii). Since $D_n^s = D_n$ for all $s \in S$, all projections D_n belong to the center of $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is the direct product of the algebras $\mathfrak{B}D_n$. Since the direct product of hyperfinite algebras is, of course, hyperfinite, it is sufficient to show that each $\mathfrak{B}D_n$ is hyperfinite. Clearly,

$$\mathfrak{B}D_n = \{\mathfrak{A}D_n, \{U_sD_n\}_{s\in S}\}'' = \{\mathfrak{A}D_n, \{U_sD_n\}_{s\in S_1}\}'',$$

where S_1 is a subgroup of $\operatorname{aut}\mathfrak{Z}(\mathfrak{B}D_n)$.

In the case $n > 0$, the algebra $\mathfrak{B}D_n$ is hyperfinite, since all hypotheses of Lemma 2 are satisfied.

If $n = 0$, then it follows from (ii) that $[S_1] = [K]$, where

$$K = \bigcup_{m=1}^{\infty} K_m, \quad K_m = (Z_2)^m.$$

By Lemma 1, the algebra $\mathfrak{B}D_0$ consists of elements of the form $\sum_k a_k U_k$, where $a_k \in \mathfrak{A}D_0$ and U_k is unitary in the algebra $\mathfrak{B}D_0$ for all $k \in K$. Let $\mathfrak{B}_m = \{\mathfrak{A}D_0, \{U_k\}_{k\in K_m}\}''$. Then

$$\mathfrak{B}_m \subset \mathfrak{B}_{m+1} \subset \Big(\bigcup_{m=1}^{\infty} \mathfrak{B}_m\Big)'' = \mathfrak{B}D_0.$$

Since K_m is finite, we can apply (iv) and then Lemma 2. Therefore, $\mathfrak{B}_m$ is hyperfinite for all $m \geqslant 1$, and so is $\mathfrak{B}D_0$. Thus Lemma 3 follows.

Proof of the Theorem. Let G be a countable nilpotent group of unitaries in a separable Hilbert space. Let

$$G = G_1 \supset G_2 \supset \ldots \supset G_n \supset G_{n+1} = 1$$

be the lower central series of G. We let

$$A = \begin{cases} \text{a maximal Abelian subgroup of } G & \text{if } n \leqslant 2, \\ G_{n-1} & \text{otherwise.} \end{cases}$$

Then the centralizer $C(A)$ of A in G is a nilpotent group of class less than or equal to $n-1$ and the factor group $G/C(A)$ is Abelian.

Consider a von Neumann algebra G''. We have $G'' = \{C(A)'', \{U_s\}_{s\in S}\}''$, where S is the group of all automorphisms s of A of the form $a^s = g^{-1}ag$ for some $g \in G$, and U_s are chosen from G in such a way that $a^s = U_s^{-1}aU_s$ for $a \in A$, $s \in S$ and $U_1 = 1$. Since $U_s^{-1}C(A)''U_s = C(A)''$ and $A'' \subset \mathfrak{Z}(C(A)'')$, S becomes an Abelian subgroup of automorphisms of the center $\mathfrak{Z}(C(A)'')$ of the algebra $C(A)''$. The theorem follows now from Lemma 3 by induction on the class of nilpotency n of the group G.

Remark. Using the same arguments one can prove that the Theorem holds for a countable group G, which contains an Abelian subgroup A such that the centralizer $C(A)$ of A in G is nilpotent and the quotient $G/C(A)$ is Abelian. For example, the group $ax+b$, where a and b are rationals, is so.

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