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ON A CLASS OF DIFFERENTIAL LINEAR EQUATIONS WHICH CAN BE SOLVED BY ALGEBRAICAL MEANS

BY

T. IWIŃSKI (WARSAW)

1. As it is well known the problem of solving a differential linear equation with constant coefficients can be reduced to an algebraical problem. It can be shown that there exist other linear differential equations which have the same property: their solutions can be derived from a characteristic equation constructed in the same way as in that simpler case.

2. Let D denote the operator of differentiation and $D^0 = I$ the identity operator. We shall consider a linear differential equation of the form

$$P_n(D)y = \sum_{i=0}^{n} a_i D^i y = f(x). \tag{1}$$

We assume that the functions a_i are continuous and, in addition, that $a_n \equiv 1$.

The equation in r

$$P_n(r) = \sum_{i=0}^{n} a_i r^i = 0 \tag{2}$$

is the characteristic equation corresponding to equation (1).

THEOREM. *If the characteristic equation* (2) *of the linear differential equation* (1) *has $n-1$ constant solutions, then the general solution of equation* (1) *may be defined in terms of the full sample of its solutions* (*we shall denote them by r_i, $i = 1, 2, \ldots, n$*).

Proof. Let $r_i = \text{const}$ for $i = 1, 2, \ldots, n-1$, and $r_n = r(x)$. Under these assumptions the product

$$(D-r_nI)[(D-r_{n-1}I)\ldots(D-r_1I)]$$

is identical with the operator $P_n(D)$, which can be easily verified (this follows from the fact that the only non-constant r_i is in the first factor). This means that the operator $P_n(D)$ can be factorized into linear factors.

It follows from Theorem 5 of [1] that if the left-side expression of (1) can be factorized into linear operator factors, then equation (1) can be easily solved. Applying formulas (3.12) of [1] we come to the following conclusions:

1° If we pose $f \equiv 0$ in (1), then the $n-1$ linearly independent solutions of that equation corresponding to the constants r_i $(i = 1, 2, \ldots \ldots, n-1)$ will take the same form as in the case of an equation with constant coefficients.

2° The last linearly independent solution will take the form

$$y_n = e^{r_1 x} \int e^{(r_2 - r_1)x} \int \ldots \int e^{(r_{n-1} - r_{n-2})x} \int e^{R_n(x) - r_{n-1}x} dx^{n-1}, \tag{3}$$

where

$$R_n = \int r_n(x)\, dx.$$

3° The function

$$Y_n = e^{r_1 x} \int e^{(r_2 - r_1)x} \int \ldots \int e^{R_n - r_{n-1}x} \int f e^{-R_n} dx^n \tag{4}$$

is a particular solution of the non-homogeneous equation (1).

3. Examples. The equations

$$y'' + f(x) y' - a[a + f(x)] y = g(x), \tag{5}$$

where

$$a = \text{const} \qquad (r_1 = a,\ r_2 = -[f(x) + a]);$$

$$y''' + f y'' - [(a^2 + ab + b^2) + (a+b) f] y' + ab(a+b+f) y = g, \tag{6}$$

where

$$a, b = \text{const}, \quad f = f(x), \quad g = g(x) \quad (r_1 = a,\ r_2 = b,\ r_3 = -(f + a + b))$$

are examples of equations of the considered type. In particular, we obtain from equation (5) the two useful equations

$$y'' + y' \sin^2 x - y \cos^2 y = f(x) \quad (r_1 = -1,\ r_2 = \cos^2 x),$$
$$y'' - x y' + (x-1) y = f(x) \quad (r_1 = 1,\ r_2 = x - 1).$$

From equation (6) we can obtain a family of equations with left-hand side expression depending only on one function $f(x)$:

$$y''' + f(x) y'' - y' - f(x) y = g(x) \quad (r_1 = 1,\ r_2 = -1,\ r_3 = -f(x)).$$

Many other examples of equations of the type considered here can be found in papers and books devoted to differential equations. For example, the equations 2.45, 2.77, 2.111, 2.112, 2.119, 2.129, 2.133, 2.136, 2.141, 2.158, 2.192, 2.262, 2.264, 2.304a (with $a = -2$), and 3.78 in the well-known monography [2] of Kamke can be solved by an application of our method. Many other examples can be found in technical applications.

REFERENCES

[1] T. Iwiński, *The generalized equations of Riccati and their applications to the theory of linear differential equations,* Rozprawy Matematyczne 23, Warszawa 1961.

[2] E. Kamke, *Differentialgleichungen,* Band 1, Leipzig 1959.

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