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A CHARACTERIZATION OF THE BALL

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1. Suppose we have a convex body and a level board with a circular hole in it. Suppose that in whichever way we place the body above the board, we can translate it by parallel motion to a position above the hole so that, if dropped from there, it will fit exactly. Is the body a ball?

This problem was suggested to one of the authors by Hugo Steinhaus. The purpose of this note is to give an affirmative answer. To check whether a given convex body of uniform density is a ball, we suggest the following test: The body is lowered on a string which moves along the axis of symmetry of the hole in the board, and the string is under observation. The test will be called successful if the string does not move sideways before slackening and after it slackens, the body fits the hole exactly (non-fitting can be detected by putting a source of light under the board). The body is a ball if and only if each such test is successful.

In the proof we shall find it more convenient to fix the body and move the board. The mathematical model for the problem is as follows. We consider the cylinder $\mathscr{C}$

$$\mathscr{C} = \{(x, y, z) | x^2+y^2 \leqslant 1,\ z \geqslant 0\}$$

in $\mathscr{E}^3$ and we associate with $\mathscr{C}$ the sets

$$\overset{0}{\mathscr{C}} = \{(x, y, z) | x^2+y^2 < 1,\ z > 0\} = \text{the interior of } \mathscr{C},$$

$$\mathscr{D} = \{(x, y, 0) | x^2+y^2 \leqslant 1\} = \text{the base of } \mathscr{C},$$

$$\overset{0}{\mathscr{D}} = \{(x, y, 0) | x^2+y^2 < 1\} = \text{the interior of } \mathscr{D},$$

$$\mathscr{K} = D - \overset{0}{D} = \text{the base circle of } \mathscr{C},$$

$$\alpha = \{(x, y, 0)\} = \text{the base plane of } \mathscr{C}.$$

Every image $\tau\mathscr{C}$ of $\mathscr{C}$ by an isometry τ will be called a *directed cylinder* with interior $\tau\overset{0}{\mathscr{C}}$, base $\tau\mathscr{D}$, interior of base $\tau\overset{0}{\mathscr{D}}$, base circle $\tau\mathscr{K}$

and base plane $\tau\alpha$. With any directed cylinder $\mathscr{C}'$ we shall associate its direction $d\mathscr{C}'$ which is a point on the sphere

$$\mathscr{S}^2 = \{(x, y, z) | x^2+y^2+z^2 = 1\}$$

defined as follows: If $\mathscr{C}' = \tau\mathscr{C}$ where τ is a rotation about $(0, 0, 0)$, then $d\mathscr{C}'$ is the centre of the hemisphere $\mathscr{C}' \cap \mathscr{S}^2$; any cylinder obtained from $\mathscr{C}'$ by translation has, by definition, the same direction as $\mathscr{C}'$.

Now let $\mathscr{B}$ be a closed bounded convex set and let $\mathscr{C}$ be a directed cylinder with base $\mathscr{D}$ and base plane α. We shall say that $\mathscr{C}$ *sits on* $\mathscr{B}$ if

(*) $\mathscr{D} \subset \mathscr{B}$,

(**) *every point of $\mathscr{B}$ which lies on the same side of α as $\overset{0}{\mathscr{C}}$ belongs to $\mathscr{C}$.*

There is no further assumption about those points of $\mathscr{B}$ which lie on α. The positive answer to the question of Steinhaus is obviously a consequence of the following

THEOREM. *If for every direction d_0 there exists at least one directed cylinder $\mathscr{C}_0$ of that direction (i. e. with $d\mathscr{C}_0 = d_0$) which sits on $\mathscr{B}$, then $\mathscr{B}$ is a ball.*

The proof will be prepared in sections 2 and 3, and its final part will be given in section 4.

2. We introduce some new notions. If $\mathscr{C}$ is a directed cylinder with base plane α, then every point of $\mathscr{E}^3$ which is on the same (opposite) side of α as $\overset{0}{\mathscr{C}}$ will be said *to be above* (*below*) α. A set of points will be said *to lie above* (*below*) α if every point of the set lies above (below) α.

If $\mathscr{C}_1, \mathscr{C}_2$ are directed cylinders with base circles $\mathscr{K}_1, \mathscr{K}_2$ and base planes α_1, α_2, then we shall say that $\mathscr{C}_1$ and $\mathscr{C}_2$ *meet properly* if

(a) $\mathscr{K}_1 \cap \mathscr{K}_2$ is a two-point set,

(b) if the two arcs in which $\mathscr{K}_1 \cap \mathscr{K}_2$ divides $\mathscr{K}_i$ have different lengths, then the greater of these arcs does not lie above α_j $(i, j = 1, 2;$ $i \neq j)$.

To illustrate this notion, let $\mathscr{C}$ be the directed cylinder $\{(x, y, z) | x^2 + y^2 \leqslant 1,\ z \geqslant 0\}$ and let μ be a translation in the direction of the positive x-axis such that $\mu\mathscr{K} \cap \mathscr{K}$ has two points. If ν is a rotation about the common chord of $\mu\mathscr{K}$ and $\mathscr{K}$ with rotation angle smaller than $\pi/2$, then $\mathscr{C}$ and $\nu\mu\mathscr{C}$ meet properly if and only if ν takes the positive z-axis towards the positive x-axis.

The main result of this section is

LEMMA 1. *If $\mathscr{C}_1, \mathscr{C}_2$ are two different directed cylinders sitting on $\mathscr{B}$ and $\overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}}_2 \neq \emptyset$, then $\mathscr{C}_1$ and $\mathscr{C}_2$ meet properly.*

The proof is based on auxiliary propositions (A), (B) and (C). The base, its interior, the base circle and base plane of $\mathscr{C}_i$ will be denoted by $\mathscr{D}_i$, $\overset{0}{\mathscr{D}}_i$, $\mathscr{K}_i$ and a_i respectively. We assume throughout that $\mathscr{C}_1$, $\mathscr{C}_2$ both sit on $\mathscr{B}$ and $\overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}}_2 \neq \emptyset$.

(A) *We cannot have* $\overset{0}{\mathscr{D}}_1$ *below* α_2 *and simultaneously* $\overset{0}{\mathscr{D}}_2$ *below* a_1.

Proof. We assume that $\overset{0}{\mathscr{D}}_1$ lies below a_2 and $\overset{0}{\mathscr{D}}_2$ lies below a_1. It is clear that the base planes a_1, a_2 cannot be parallel; let $m = a_1 \cap a_2$ be the line which they have in common. Let a_i^* be the open halfplane (not containing m) equal to that part of a_i which lies above a_j $(i \neq j)$. Thus

$$(\overset{0}{\mathscr{D}}_1 \cup \overset{0}{\mathscr{D}}_2) \cap (a_1^* \cup a_2^*) = \emptyset.$$

Let $\mathscr{P}$ be the open subset of $\mathscr{E}^3$ consisting of all those points which lie simultaneously above a_1 and above a_2. It is clear that $\mathscr{P}$ is one of the four regions into which $\mathscr{E}^3$ is partitioned by the planes a_1, a_2, moreover

$$\operatorname{Fr}\mathscr{P} = a_1^* \cup a_2^* \cup m.$$

If for any point $P \in \mathscr{P}$ we denote by P_0 the point in $\operatorname{Fr}\mathscr{P}$ which is nearest to P (or one of these points, if there are more), then clearly $P_0 \in a_1^* \cup a_2^*$. Now let $P \in \overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}}_2$. Then $P \in \mathscr{P}$ and $P_0 \in a_1^* \cup a_2^*$. But P_0 is the orthogonal projection of P into a_1 or a_2, and from $P \in \overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}}_2$ it follows that this vertical projection is either in $\overset{0}{\mathscr{D}}_1$ or in $\overset{0}{\mathscr{D}}_2$ and hence certainly not in $a_1^* \cup a_2^*$. This contradiction shows that we cannot have $\overset{0}{\mathscr{D}}_1$ below a_2 and $\overset{0}{\mathscr{D}}_2$ below a_1.

(B) *If either of the sets* $\mathscr{K}_1 \cap a_2$, $\mathscr{K}_2 \cap a_1$ *is a two-point set, then so is the other,*

$$\mathscr{K}_1 \cap a_2 = \mathscr{K}_2 \cap a_1 = \mathscr{K}_1 \cap \mathscr{K}_2,$$

and if the two arcs in which $\mathscr{K}_1 \cap \mathscr{K}_2$ *divides* $\mathscr{K}_i$ *have different lengths, then the smaller of these arcs lies above* a_j $(i \neq j)$.

Proof. Let PQ denote the segment of the straight line joining points P and Q. We show first the following implication

(b) if $\mathscr{K}_i \cap a_j = \{P_i, Q_i\}$ and $P_i \neq Q_i$, then $\mathscr{K}_j \cap a_i = \{P_j, Q_j\}$, $P_j \neq Q_j$ and $P_iQ_i \subset P_jQ_j$.

Let $\mathscr{K}_i \cap a_j = \{P_i, Q_i\}$; $P_i \neq Q_i$. Since $\mathscr{K}_i$ is not a subset of a_j, there are points of $\mathscr{K}_i$ which lie above a_j and are arbitrarily close to P_i and Q_i respectively. From $\mathscr{K}_i \subset \mathscr{B}$ and by condition (**), it follows that $P_i, Q_i \in \mathscr{C}_j \cap a_j = \mathscr{D}_j$. Let m denote the line $a_1 \cap a_2$. We have $P_i, Q_i \in m$ which implies that $m \cap \mathscr{D}_j \neq \emptyset$ and therefore $\mathscr{K}_j \cap a_i = \mathscr{K}_j \cap m$ is a two point set, say $\{P_j, Q_j\}$; $P_j \neq Q_j$, and moreover $P_iQ_i \subset P_jQ_j$. This proves (b).

Now, to prove the first part of (B), we assume without loss of generality that $\mathscr{K}_1 \cap a_2 = \{P_1, Q_1\}$; $P_1 \neq Q_1$. Then, applying (b) with $i = 1$, $j = 2$, we have $\mathscr{K}_2 \cap a_1 = \{P_2, Q_2\}$ and $P_1Q_1 \subset P_2Q_2$. Applying (b) again, with $i = 2$, $j = 1$, we obtain $P_2Q_2 \subset P_1Q_1$, whence $\{P_1, Q_1\} = \{P_2, Q_2\}$. Therefore $\mathscr{K}_1 \cap a_2 = \mathscr{K}_2 \cap a_1 = \mathscr{K}_1 \cap \mathscr{K}_2$.

To complete the proof of (B), we shall show that if $\mathscr{K}_1^*$ denotes the greater of the two open arcs into which $\mathscr{K}_1$ is split by $\mathscr{K}_1 \cap \mathscr{K}_2$, then $\mathscr{K}_1^*$ does not lie above a_2. Assuming that $\mathscr{K}_1^*$ lies above a_2, we consider the diameter chord Δ of $\mathscr{K}_1$ which is parallel to P_1Q_1. Obviously the two-point set $\Delta \cap \mathscr{K}_1$ is in $\mathscr{K}_1^*$, thus Δ lies above a_2. It follows from (**) that $\Delta \subset \mathscr{C}_2$ and therefore the orthogonal projection of Δ into a_2 is a diameter of $\mathscr{K}_2$. If follows that the centre F_1 of $\mathscr{K}_1$ projects on the centre F_2 of $\mathscr{K}_2$. Since the centre of a circle of a given radius is equidistant from all chords which have a given length, the distance of F_1 from P_1Q_1 is the same as the distance of F_2 from P_1Q_1. Thus F_1F_2 is the base of a triangle whose other sides lie in a_1 and a_2 and have equal lengths. It follows that F_1F_2 is not perpendicular to a_2 contradicting our previous observation that F_2 is the projection of F_1. The contradiction shows that $\mathscr{K}_1^*$ lies below a_2. In the same way we show that $\mathscr{K}_2^*$ must lie below a_1.

(C) $\overset{0}{\mathscr{D}}_i$ *cannot lie above* a_j $(i, j = 1, 2)$.

Proof. Suppose that $\overset{0}{\mathscr{D}}_2$ is above a_1. Then, by (**), $\overset{0}{\mathscr{D}}_2 \subset \mathscr{C}_1$ and so $\mathscr{K}_2 \subset \mathscr{C}_1$. $\tilde{\mathscr{K}}_2$, the orthogonal projection of $\mathscr{K}_2$ into a_1 is an ellipse, with unit major semi-axis, contained in $\mathscr{D}_1$. Thus $\tilde{\mathscr{K}}_2$ meets $\mathscr{K}_1$ at the ends of the major axis. Hence $\mathscr{K}_2$ meets the boundary of $\mathscr{C}_1$ in two points R and S, such that the segment RS is parallel to a_1. $\overset{0}{\mathscr{D}}_2 \cap a_1 = \emptyset$ implies that R and S lie above a_1.

Let R' be the point of $\mathscr{K}_1$ such that RR' is perpendicular to a_1, and let γ be the plane containing RR' that is tangent to $\mathscr{K}_1$. A point x will be said to be *below* (*above*) γ if x and S are (are not) on the same side of γ. Suppose $x \in \mathscr{B}$. As $\mathscr{B}$ is convex, $Rx \subset \mathscr{B}$. But all points of $\mathscr{B}$ near to R lie in $\mathscr{C}_1$ and Rx contains points arbitrarily near to R. Hence x is either below γ, or x, R and R' are collinear.

Now consider the cylinder $\mathscr{C}$ that sits on $\mathscr{B}$, has direction perpendicular to γ and which has points above γ. Let $\mathscr{D}$, $\mathscr{K}$ and a be, as usual, the base, base circle and base plane of $\mathscr{C}$. $\mathscr{K}$ must lie below γ, as the only points of $\mathscr{B}$ not below γ lie on the line through R and R'. Hence, by (**), $R \in \mathscr{C}$. $\mathscr{K}$ thus has points above a_1 and hence $\mathscr{D} \cap \overset{0}{\mathscr{C}}_1 \neq \emptyset$ (using (**) on $\mathscr{C}_1$). This implies that $a \cap \mathscr{K}_1$ is a pair of points and so is $a \cap \mathscr{K}_2$.

By (B), $\mathscr{K}$ meets $\mathscr{K}_1$ and $\mathscr{K}_2$ each in two points. Let $\mathscr{K} \cap \mathscr{K}_1 = \{P_1, Q_1\}$ and $\mathscr{K} \cap \mathscr{K}_2 = \{P_2, Q_2\}$. Let l_1 be the length of the arc of $\mathscr{K}$ which joins P_1 to Q_1 and lies above a_1. By (B), $l_1 \leqslant \pi$. The segments P_1Q_1 and P_2Q_2 are at the same distance from γ, γ is perpendicular to a_1 and a_2, and γ is tangent to the circles $\mathscr{K}_1$ and $\mathscr{K}_2$. Hence the segments P_1Q_1 and P_2Q_2 are of equal length. Thus, if l_2 is the length of the arc of $\mathscr{K}$ which is above a_1 and which joins P_2 and Q_2, either $l_2 = l_1$ or $l_2 = 2\pi - l_1$. But as P_2 or Q_2 is above a_1, $l_2 < l_1$ and so $2\pi - l_1 < l_1$. This contradicts the fact that $l_1 \leqslant \pi$. We have therefore shown that $\overset{0}{\mathscr{D}}_2$ does not lie above a_1, and similarly $\overset{0}{\mathscr{D}}_1$ does not lie above a_2.

Proof of Lemma 1. If $a_1 = a_2$, then $\mathscr{K}_1$ and $\mathscr{K}_2$ are circles in the same plane which do not coincide (since $\mathscr{C}_1 \neq \mathscr{C}_2$, and $\overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}}_2 \neq \emptyset$), thus $\mathscr{K}_1 \cap \mathscr{K}_2$ has two points. Clearly none of the arcs in which $\mathscr{K}_i$ is divided by $\mathscr{K}_1 \cap \mathscr{K}_2$ lies above a_j. If $a_1 \neq a_2$, then these planes cannot be parallel since otherwise we arrive at a contradiction with (C). Let m denote the line $a_1 \cap a_2$. This line divides each a_i in two open halfplanes (neither containing m); let a_i^* be the one above a_j and a_i' the one below a_j $(i \neq j)$. Suppose for a moment that

$$\overset{0}{\mathscr{D}}_1 \cap m = \overset{0}{\mathscr{D}}_2 \cap m = \emptyset.$$

Then one of the following four possibilities must occur:

(i) $\overset{0}{\mathscr{D}}_1 \subset a_1^*$ and $\overset{0}{\mathscr{D}}_2 \subset a_2^*$,

(ii) $\overset{0}{\mathscr{D}}_1 \subset a_1^*$ and $\overset{0}{\mathscr{D}}_2 \subset a_2'$,

(iii) $\overset{0}{\mathscr{D}}_1 \subset a_1'$ and $\overset{0}{\mathscr{D}}_2 \subset a_2^*$,

(iv) $\overset{0}{\mathscr{D}}_1 \subset a_1'$ and $\overset{0}{\mathscr{D}}_2 \subset a_2'$.

But each of these contradicts either (A) or (C). We have shown that $\overset{0}{\mathscr{D}}_i \cap m = \emptyset$ implies $\overset{0}{\mathscr{D}}_j \cap m \neq \emptyset$ which in turn implies that $\mathscr{K}_j \cap a_i$ is a two-point set. Thus the assertion of Lemma 1 follows from (B).

3. We shall say that a cylinder $\mathscr{C}$ *sits flat* on $\mathscr{B}$ if $\mathscr{C}$ sits on $\mathscr{B}$ and $\mathscr{B} \cap \overset{0}{\mathscr{C}} = \emptyset$. By the *angle* between two directions we shall mean the length of the shortest arc on the unit sphere joining these directions. Our second lemma is as follows:

LEMMA 2. *Suppose that $\mathscr{C}$ sits on $\mathscr{B}$ and it does not sit flat. Then there exists a point Q above a and on the axis of symmetry of $\mathscr{C}$ with the following property: For every direction d' which is orthogonal to $d\mathscr{C}$ there exist exactly two directed cylinders $\mathscr{C}_1, \mathscr{C}_2$ sitting on $\mathscr{B}$ such that*

(a) *each of the base circles* $\mathscr{K}_1$, $\mathscr{K}_2$ *passes through* Q *and cuts* $\mathscr{K}$ *at two points*;

(b) *each of the directions* $d\mathscr{C}_1$, $d\mathscr{C}_2$ *is orthogonal to* d'.

(c) *the angle between* $d\mathscr{C}_i$ *and* $d\mathscr{C}$ $(i = 1, 2)$ *is at most* $\pi/2$.

The proof is based on auxiliary propositions (D), (E), ..., (K).

(D) *Let* $\mathscr{C}$ *sit on* $\mathscr{B}$ *and suppose that* $\mathscr{C}$ *does not sit flat. Then every point* P *in* $\overset{0}{\mathscr{D}}$ *is the orthogonal projection on* α *of a unique point* $P_1 \in \operatorname{Fr}\mathscr{B}$ *which lies above* α.

Proof. To prove the uniqueness of P_1, assume that $P_1, P_2 \in \operatorname{Fr}\mathscr{B}$ are both above α and project on P. Then one of them, say P_1, is higher above α than P_2. The cone with vertex P_1 and base $\mathscr{D}$ is contained in $\mathscr{B}$ and P_2 is in the interior of this cone. Hence $P_2 \in \overset{0}{\mathscr{B}}$, a contradiction.

To prove the existence of P_1, note that $\mathscr{C} \cap \overset{0}{\mathscr{B}} \neq \emptyset$ implies the existence of some $R \in \mathscr{B}$ which lies above α. The cone with vertex R and base $\mathscr{D}$ is contained in $\mathscr{B}$ and above every point P in $\overset{0}{\mathscr{D}}$ there are points of this cone. Hence, as $\mathscr{B}$ is bounded, above every P in $\overset{0}{\mathscr{D}}$, there is some $P_1 \in \operatorname{Fr}\mathscr{B}$.

A direction $d \in \mathscr{S}^2$ will be called *regular* if there exists exactly one cylinder $\mathscr{C}$ of that direction sitting on $\mathscr{B}$.

(E) *If* $\mathscr{C}$ *sits on* $\mathscr{B}$ *and* $\mathscr{C}$ *does not sit flat, then* $d\mathscr{C}$ *is regular.*

Proof. Suppose $d\mathscr{C}$ is not regular and denote by $\mathscr{C}'$ another cylinder which sits on $\mathscr{B}$ and has direction $d\mathscr{C}$. The base planes α, α' of $\mathscr{C}$ and $\mathscr{C}'$ are parallel and since none of them can lie above the other (see (C)), they coincide. The bases $\mathscr{D}$ and $\mathscr{D}'$ are different, for otherwise we would have $\mathscr{C} = \mathscr{C}'$. Let $P \in \overset{0}{\mathscr{D}} - \mathscr{D}'$. As there are no points of $\mathscr{B}$ outside $\mathscr{C}'$ and above α, P is not an orthogonal projection on α of any point $P_1 \in \mathscr{B}$ which lies above α. By (D), this contradicts our assumption that $\mathscr{C}$ does not sit flat on $\mathscr{B}$.

(F) *The set of all non-regular directions is finite.*

Proof. Let $\{d_\tau\}_{\tau \in T}$ be the set of all non-regular directions. It follows from (E) that we can associate with every d_τ a cylinder $\mathscr{C}_\tau$ which sits flat on $\mathscr{B}$, such that $d\mathscr{C}_\tau = d_\tau$. Let us show that $\tau \neq \mu$ implies $\overset{0}{\mathscr{C}}_\tau \cap \overset{0}{\mathscr{C}}_\mu = \emptyset$ (every $\tau, \mu \in T$). Assuming the contrary, we have, by Lemma 1 that $\mathscr{C}_\tau$ and $\mathscr{C}_\mu$ meet properly. Then $\mathscr{K}_\tau \cap \mathscr{K}_\mu$ is a two-point set which shows that the discs $\overset{0}{\mathscr{D}}_\tau$ and $\overset{0}{\mathscr{D}}_\mu$ intersect. They obviously do not lie in the same plane (being orthogonal to different directions) which implies that $\overset{0}{\mathscr{D}}_\tau \cap \overset{0}{\mathscr{C}}_\mu \neq \emptyset$, whence $\mathscr{B} \cap \overset{0}{\mathscr{C}}_\mu \neq \emptyset$, contradicting the assumption that $\mathscr{C}_\mu$ sits flat.

Now let us consider an open ball $\tilde{\mathscr{B}}$ which contains $\mathscr{B}$. Let δ_0 be the minimum of all distances $\delta(P, Q)$ where $P \in \mathscr{B}$ and $Q \in \mathrm{Fr}\, \tilde{\mathscr{B}}$. It is clear that if $\mathscr{C}$ is a cylinder sitting flat on $\mathscr{B}$, then the volume of $\overset{0}{\mathscr{C}} \cap \tilde{\mathscr{B}}$ is greater than $\pi\delta_0$ (the base of $\overset{0}{\mathscr{C}} \cap \tilde{\mathscr{B}}$ is a unit disc). By the above, the sets $\overset{0}{\mathscr{C}}_\tau \cap \tilde{\mathscr{B}}$ are all pairwise disjoint. Since each of them has volume greater than $\pi\delta_0$ and they are all contained in $\tilde{\mathscr{B}}$, we conclude that T is finite.

From now on we shall fix our coordinate system so that the cylinder $\mathscr{C}$ assumed in Lemma 2 is $\{(x, y, z) | x^2+y^2 \leqslant 1, z \geqslant 0\}$ and the direction d' is on the positive part of the z-axis.

Let $\mathscr{G}$ be the group of all orientation preserving isometries of $\mathscr{E}^3$. To define a topology in $\mathscr{G}$, we note that $\mathscr{G} = \mathscr{E}^3 \Theta_3$, i. e. $\mathscr{G}$ is the semi-direct product of the subgroup Θ_3 of all rotations about $(0, 0, 0)$ (which we identify with the group Θ_3 of orthogonal 3×3 matrices with determinant 1) and the subgroup of all translations (which we identify with $\mathscr{E}^3$). There is thus a natural bijection $\mathscr{G} \leftrightarrow \mathscr{E}^3 \times \Theta_3$ which induces a topology in $\mathscr{G}$ from the product topology in $\mathscr{E}^3 \times \Theta_3$. It is well known that in this topology, the mapping

$$a: \mathscr{G} \times \mathscr{E}^3 \to \mathscr{E}^3,$$

given by $a(g, P) = gP$, is continuous.

(G) *Let $\mu: I \to \mathscr{G}$ be a continuous mapping of the unit interval I into $\mathscr{G}$. Denoting by μ_t the image of t, we assume that the isometry μ_0 leaves $\mathscr{K}$ invariant,* i. e. *$\mu_0 \mathscr{K} = \mathscr{K}$, and that the plane of the circle $\mu_1 \mathscr{K}$ is parallel to the z-axis. Then there exists a t_1 in I such that $\mu_{t_1}\mathscr{K}$ cuts the z-axis.*

Proof. We denote by $F: I \times \mathscr{K} \to \mathscr{E}^3$ the composition of the continuous maps

$$I \times \mathscr{K} \xrightarrow{\mu\times\varepsilon} \mathscr{G} \times \mathscr{E}^3 \xrightarrow{a} \mathscr{E}^3,$$

where $\varepsilon: \mathscr{K} \to \mathscr{E}^3$ is the inclusion map. For every t in I, let $F_t: \mathscr{K} \to \mathscr{E}^3$ be given by $F_t(P) = F(t, P)$ for all $P \in \mathscr{K}$. Then

$$F_0(\mathscr{K}) = \mu_0 \mathscr{K} = \mathscr{K} \quad \text{and} \quad F_1(\mathscr{K}) = \mu_1 \mathscr{K}.$$

As the plane of $\mu_1 \mathscr{K}$ is parallel to the z-axis, $\mu_1 \mathscr{K}$ either meets the z-axis, or $\mu_1 \mathscr{K}$ bounds a disc that does not meet the z-axis. Suppose the latter is true. Then $F_1: \mathscr{K} \to \mathscr{E}^3$ is null-homotopic in $\mathscr{E}^3 - \{(0, 0, z)\}$ whereas $F_0: \mathscr{K} \to \mathscr{E}^3$ represents a generator of the (non-trivial) fundamental group of $\mathscr{E}^3 - \{(0, 0, z)\}$. It follows that F_1 and F_0 are not homotopic in $\mathscr{E}^3-\{(0,0,z)\}$. F_t is a homotopy sending F_0 to F_1, in $\mathscr{E}^3$, hence $F(I\times\mathscr{K})$ must meet the z-axis. Thus there exists a t_1 in I with the property that $F_{t_1}(\mathscr{K}) = \mu_{t_1}\mathscr{K}$ cuts the z-axis.

Let us denote by Ω the set of all cylinders $\mathscr{C}'$ which sit on $\mathscr{B}$ whose directions $d\mathscr{C}'$ form angles not greater than $\pi/2$ with $(0,0,1)$.

(H) *There exists a* 1-1 *mapping* $\omega:\Omega\to\mathscr{G}$ *which associates with every cylinder* $\mathscr{C}'\in\Omega$ *an isometry* $\omega(\mathscr{C}')$ *such that*

1) $\mathscr{C}' = \omega(\mathscr{C}')\mathscr{C}$ (i. e. $\mathscr{C}'$ *is the image of* $\mathscr{C}$ *by* $\omega(\mathscr{C}')$)

and

2) *the set* $\omega(\Omega) = \{\omega(\mathscr{C}')|\mathscr{C}'\in\Omega\}$ *is a compact subset of* $\mathscr{G}$.

Proof. Let $\Gamma_0\subset\mathscr{G}$ be the set of all those isometries $\tau\in\mathscr{G}$ which are of the form $\tau = \nu\varrho$, where $\nu\in\mathscr{E}^3$ and ϱ is a rotation through an angle at most $\pi/2$ about an axis through $(0,0,0)$ in the x,y-plane. It is clear that for every $\mathscr{C}'\in\Omega$, there is a unique isometry $\omega(\mathscr{C}')\in\Gamma_0$ such that $\mathscr{C}' = \omega(\mathscr{C}')\mathscr{C}$. Thus a mapping $\omega:\Omega\to\mathscr{G}$ is defined and it satisfies 1). It is immediate that ω is 1-1.

To prove that $\omega(\Omega)$ is compact, consider the set $\Gamma_1\subset\mathscr{G}$ given by

$$\Gamma_1 = \{\tau\in\mathscr{G}|\tau\mathscr{C}\in\Omega\}.$$

Evidently $\omega(\Omega) = \Gamma_0\cap\Gamma_1$, and since Γ_0 is obviously closed, it will be sufficient to show that Γ_1 is compact. We consider the auxiliary set $\Gamma_2 = \{\tau\in G|\tau(0,0,0)\in\mathscr{B}\}$ which is equal to $\mathscr{B}\Theta_3$ (note that $\mathscr{B}\subset\mathscr{E}^3\subset\mathscr{G}$) and so is compact, as it is homeomorphic to a product of two compact sets. But $\Gamma_1\subset\Gamma_2$ and Γ_1 is closed, for if we have a sequence $\tau_1,\tau_2,\tau_3,\ldots$ of isometries such that $\tau_i\mathscr{C}\in\Omega$ which converges to τ, then it is easily seen from (*) and (**) that $\tau\mathscr{C}$ sits on B and the angle between $d\tau\mathscr{C}$ and the positive z-axis is at most $\pi/2$, whence $\tau\mathscr{C}\in\Omega$. Thus Γ_1 is compact.

Let us consider the mapping $d:\Omega\to\mathscr{S}^2$ which associates with every cylinder its direction. It is clear that the set $d\Omega = \{d\mathscr{C}'|\mathscr{C}'\in\Omega\}$ is identical with the upper hemisphere.

$$\overset{+}{\mathscr{S}^2} = \{(x,y,z)|x^2+y^2+z^2 = 1,\ z\geqslant 0\}.$$

If we define the topology in Ω so that the map ω above is a homeomorphism, then Ω is a compact space, moreover we have

(I) *The mapping* $d:\Omega\to\overset{+}{\mathscr{S}^2}$ *is continuous; and if* $\mathscr{F}$ *denotes the finite set of all non-regular directions* (see (C)), *then the inverse mapping*

$$d^{-1}:(\overset{+}{\mathscr{S}^2}-\mathscr{F})\to\Omega$$

is continuous.

Proof. Let $\varkappa:\mathscr{G}\to\Theta_3$ be the map which associates with every isometry $\tau = \nu\varrho$, where $\nu\in\mathscr{E}$ and $\varrho\in\Theta_3$, the rotation ϱ, i. e. $\varkappa$ can be regarded as the projection of $\mathscr{G} = \mathscr{E}^3\Theta_3$ onto Θ_3. Since the topology of G is that of $\mathscr{E}^3 \times \Theta_3$, $\varkappa$ is obviously continuous. For every $\mathscr{C}' \in \Omega$, we have, by (H),

$$d\mathscr{C}' = d\big(\omega(\mathscr{C}')\mathscr{C}\big) = a\big(\varkappa(\omega(\mathscr{C}')), d\mathscr{C}\big) = a\big(\varkappa(\omega(\mathscr{C}')), (0, 0, 1)\big),$$

i. e. d is a composition of continuous maps.

Suppose that $\mathscr{F} = \{F_1, F_2, \ldots, F_n\}$. The continuity of d^{-1} on $\overset{+}{\mathscr{S}^2} - \mathscr{F}$ will follow if we show that d^{-1} is continuous on every set of the form $\overset{+}{\mathscr{S}^2} - \bigcup \mathscr{V}_i$ where $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_n$ are arbitrary open neighbourhoods of $F_1, F_2, \ldots, F_n$. Let $d^{-1}\mathscr{V}_i$ be the set $\{\mathscr{C}' \in \Omega | d\mathscr{C}' \in \mathscr{V}_i\}$, this set is open by the continuity of d. Clearly $\Omega - \bigcup d^{-1}\mathscr{V}_i$ is compact and

$$d : (\Omega - \bigcup d^{-1}\mathscr{V}_i) \to (\overset{+}{\mathscr{S}^2} - \bigcup \mathscr{V}_i)$$

is onto and 1-1. Hence, by a well-known theorem, its inverse

$$d^{-1} : (\overset{+}{\mathscr{S}^2} - \bigcup \mathscr{V}_i) \to (\Omega - \bigcup d^{-1}\mathscr{V}_i)$$

is continuous.

(J) *If all directions belonging to the intersection of $\overset{+}{S^2}$ and the yz-plane are regular, then there exist at least two cylinders $\mathscr{C}_1, \mathscr{C}_2 \in \Omega$ such that*

1) *each of the base circles $\mathscr{K}_1, \mathscr{K}_2$ cuts the z-axis,*

2) *$d\mathscr{C}_1$ and $d\mathscr{C}_2$ are in the yz-plane.*

Proof. Consider the map $f : I \to \overset{+}{\mathscr{S}^2}$ given by

$$f(t) = \left(0, \sin\frac{\pi}{2}t, \cos\frac{\pi}{2}t\right)$$

for $0 \leqslant t \leqslant 1$. Since $f(I)$ is in the yz-plane, we have that $f(I) \subset \overset{+}{\mathscr{S}^2} - \mathscr{F}$. Denote by μ: $I \to \mathscr{G}$ the composition of the continuous maps

$$I \xrightarrow{f} (\mathscr{S}^2 - \mathscr{F}) \xrightarrow{d^{-1}} \Omega \xrightarrow{\omega} \mathscr{G}.$$

Let us verify that μ satisfies the assumptions of (G). First we note that $d^{-1}f(0) = d^{-1}(0, 0, 1) = \mathscr{C}$ and thus

$$\mu_0\mathscr{C} = \omega\big(d^{-1}f(0)\big)\mathscr{C} = \omega(\mathscr{C})\mathscr{C} = \mathscr{C}$$

by (H), 1). It follows that $\mu_0\mathscr{K} = \mathscr{K}$. The cylinder $\mathscr{C}' = d^{-1}f(1) = d^{-1}(0, 1, 0)$ has direction $(0, 1, 0)$ and thus the plane of its base circle K' is parallel to the z-axis. Moreover,

$$\mu_1\mathscr{C} = \omega\big(d^{-1}f(1)\big)\mathscr{C} = \omega(\mathscr{C}')\mathscr{C} = \mathscr{C}'$$

by (H), 1), whence it follows that $\mu_1\mathscr{K} = \mathscr{K}'$. Thus by (G), there exists a t_1 in I such that the circle $\mu_{t_1}\mathscr{K}$ cuts the z-axis. Let $\mathscr{C}_1 = \mu_{t_1}\mathscr{C}$. To obtain the direction of $\mathscr{C}_1$, we note that, by (H), 1) we have

$$d^{-1}f(t) = \omega\big(d^{-1}f(t)\big)\mathscr{C} = \mu_t\mathscr{C} \qquad \text{for every } t \text{ in } I,$$

and applying this with $t = t_1$ we get $d^{-1}f(t_1) = \mu_{t_1}\mathscr{C} = \mathscr{C}_1$. Hence $\mathscr{C}_1 \in \Omega$ and $d\mathscr{C}_1 = f(t_1) \in f(I)$. Similarly, taking instead of f the mapping g: $I \to \overset{+}{\mathscr{S}^2}$ given by

$$g(t) = \left(0, -\sin\frac{\pi}{2}t, \cos\frac{\pi}{2}t\right)$$

for $0 \leqslant t \leqslant 1$, we find another cylinder $\mathscr{C}_2 \in \Omega$ such that $d\mathscr{C}_2 \in g(I)$. Since $f(I) \cap g(I) = \{(0, 0, 1)\}$, the cylinders $\mathscr{C}_1, \mathscr{C}_2$ could coincide only if they were both equal to $\mathscr{C}$, but this is obviously impossible. Clearly the directions $d\mathscr{C}_1, d\mathscr{C}_2$ are in the yz-plane.

Let us denote by Q the unique point on the positive z-axis which belongs to $\mathrm{Fr}\mathscr{B}$. The existence and uniqueness of Q follow from (D).

(K) *If $\mathscr{C}_1 \in \Omega$ is a cylinder whose base circle $\mathscr{K}_1$ cuts the z-axis, then $Q \in \mathscr{K}_1$ and $\mathscr{K}_1 \cap \mathscr{K}$ is a two-point set.*

Proof. Suppose first that $d\mathscr{C}_1$ is in the xy-plane. Then the base plane a_1 of $\mathscr{C}_1$ contains the z-axis and $\mathscr{K} \cap a_1 = \{P_1, P_2\}$ where P_1 and P_2 are diametrically opposite on $\mathscr{K}$. By (B) we have $\mathscr{K}_1 \cap \mathscr{K} = \{P_1, P_2\}$ whence it follows that $\mathscr{K}_1$ must cut the z-axis in two diametrically opposite points; one of these must be above the xy-plane, and since it belongs to $\mathrm{Fr}\mathscr{B}$ (note that $K_1 \subset \mathrm{Fr}\mathscr{B}$), it must be Q.

Now suppose that the angle between $d\mathscr{C}_1$ and the positive z-axis is less than $\pi/2$. Let R be any point of $\mathscr{K}_1$ which is on the z-axis. To prove $R = Q$, it is enough to show that R lies above the xy-plane, since obviously we have $R \in \mathrm{Fr}\mathscr{B}$.

Suppose first that $R = (0, 0, 0)$. Let s be the line through $(0, 0, 0)$ containing the direction $d\mathscr{C}_1$ and let $S \in s \cap \mathscr{C}_1$ be different from R. Since the angle $\widehat{QRS}$ is less than $\pi/2$, it is evident that $(s \cap \mathscr{C}_1) \cap \overset{0}{\mathscr{C}} \neq \emptyset$ whence $\overset{0}{\mathscr{C}_1} \cap \overset{0}{\mathscr{C}} \neq \emptyset$. It follows, by Lemma 1, that $\mathscr{C}_1$ and $\mathscr{C}$ meet properly and therefore $\mathscr{K}_1$ meets the base plane of $\mathscr{K}$ i. e. the xy-plane, only at points of $\mathscr{K}$, contrary to $(0, 0, 0) \in \mathscr{K}_1$. Thus $R \neq (0, 0, 0)$.

Assume now that R lies below the xy-plane. Let s be the line through R orthogonal to the base plane a_1 of $\mathscr{C}_1$ and let $S \in s \cap \mathscr{C}_1$ be different from R. The angle $\widehat{QRS}$ is by assumption smaller than $\pi/2$ whence the point $(0, 0, 0)$ of QR lies above a_1. But as $\overset{0}{\mathscr{D}}$ cannot lie entirely above a_1, by (C) we have $\overset{0}{\mathscr{D}} \cap a_1 \neq \emptyset$. It follows that $\mathscr{K} \cap a_1 = \{P_1, P_2\}$ where $P_1 \neq P_2$. We note that P_1, P_2 cannot be diametrically opposite points of $\mathscr{K}$, for in that case $(0, 0, 0)$ would belong to the chord P_1P_2 and thus to a_1. By (B) we deduce that $\mathscr{K}_1 \cap \mathscr{K} = \{P_1, P_2\}$ and that the smaller of the two arcs in which $\mathscr{K}$ is divided by $\mathscr{K}_1$ lies above a_1. But as the centre $(0, 0, 0)$ of K lies above a_1, this is a contradiction. Thus R does not lie below the xy-plane.

We conclude that R lies above the xy-plane, i. e. $R = Q$. This also proves that $\mathscr{K}_1 \cap \overset{0}{\mathscr{C}} \neq \emptyset$, whence $\overset{0}{\mathscr{C}}_1 \cap \overset{0}{\mathscr{C}} \neq \emptyset$. By Lemma 1, $\mathscr{K}_1 \cap \mathscr{K}$ is a two-point set.

Proof of Lemma 2. We assume first that all directions in the yz-plane are regular. Then, if Q is the point considered in (K), we have by (J) and (K) that there are at least two cylinders $\mathscr{C}_1, \mathscr{C}_2 \in \Omega$ such that conditions (a), (b) and (c) of Lemma 2 are satisfied. On the other hand, it is easy to verify that for every point Q on the positive z-axis there are at most two directed cylinders $\mathscr{C}_1, \mathscr{C}_2$ which satisfy (a), (b) and (c). Thus the existence and uniqueness of $\mathscr{C}_1$ and $\mathscr{C}_2$ are proved. Let us observe that the angle φ between $d\mathscr{C}_1$ and $d\mathscr{C}_2$ is entirely determined by the position of Q and does not depend on d'. Moreover $0 < \varphi \leqslant \pi$.

If follows from (E) that the direction of the positive z-axis is regular. Thus, from (F), we deduce that there can be only finitely many planes containing the z-axis and some non-regular direction in $\overset{+}{\mathscr{S}}{}^2$. It follows that there can be only finitely many directions d' orthogonal to $d\mathscr{C}$ for which the assumption of (J) is not satisfied, after the coordinate system is chosen so that d' is on the positive x-axis. If d' is one of these finitely many directions, then we can find a sequence $d_1', d_2', d_3', \ldots$ of directions orthogonal to $d\mathscr{C}$ such that $d = \lim_{n\to\infty} d_n'$ and for every d_n' there are cylinders $\mathscr{C}_1^n, \mathscr{C}_2^n \in \Omega$ satisfying the hypothesis of Lemma 2, with d_n' in place of d'. Since Ω is compact, we can find subsequences $\mathscr{C}_1^{n_1}, \mathscr{C}_1^{n_2}, \ldots$ and $\mathscr{C}_2^{n_1}, \mathscr{C}_2^{n_2}, \ldots$ which converge to limits $\mathscr{C}_1$ and $\mathscr{C}_2$ in Ω. As the angle between $d\mathscr{C}_1^{n_k}$ and $d\mathscr{C}_2^{n_k}$ is $\varphi > 0$ and $d : \Omega \to \overset{+}{\mathscr{S}}{}^2$ is continuous, the angle between $d\mathscr{C}_1$ and $d\mathscr{C}_2$ is also φ and therefore $\mathscr{C}_1 \neq \mathscr{C}_2$. Obviously $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy conditions (a), (b) and (c).

4. By using Lemma 2, we are now able to prove the Theorem. Let $\mathscr{C}$ be a directed cylinder which sits on $\mathscr{B}$ and does not sit flat on $\mathscr{B}$, and Q be the point of $\mathrm{Fr}\,\mathscr{B}$ on the axis of symmetry of $\mathscr{C}$ and above the base plane a of $\mathscr{C}$. As before, $\mathscr{K}$ is the base circle of $\mathscr{C}$. There is one and exactly one sphere $\mathscr{S}$, through Q and $\mathscr{K}$. Any circle which passes through Q and meets $\mathscr{K}$ in two points meets $\mathscr{S}$ in three distinct points and hence is contained in $\mathscr{S}$. Now there are at most two directed cylinders $\mathscr{C}_1, \mathscr{C}_2$ which have directions perpendicular to a given direction d' perpendicular to $d\mathscr{C}$ such that their base circles $\mathscr{K}_1, \mathscr{K}_2$ pass through Q and cut $\mathscr{K}$ in two points. By Lemma 2 two such directed cylinders $\mathscr{C}_1, \mathscr{C}_2$ always exist, and moreover they sit on $\mathscr{B}$. As $\mathscr{K}_1, \mathscr{K}_2 \subset \mathrm{Fr}\,\mathscr{B}$, it follows that every unit circle in $\mathscr{S}$, which passes through Q and cuts $\mathscr{K}$ in two points, is the base circle of some cylinder sitting on $\mathscr{B}$ and is contained in $\mathrm{Fr}\,\mathscr{B}$. Let O be the centre of $\mathscr{S}$, and ψ the semi-vertical angle of the right circular cone with vertex O and base $\mathscr{K}$. If P is any point of $\mathscr{S}$ such that the angle $\widehat{POQ}$ is less than or equal to 2ψ, there is a unit circle through P and Q which meets $\mathscr{K}$ in two points. Hence the whole spherical cap of $\mathscr{S}$, with axis OQ and which subtends a semi-angle 2ψ at O, is contained in $\mathrm{Fr}\,\mathscr{B}$.

Now let Q' be any point of $\mathscr{K}$. Let $\mathscr{K}'$ be the unit circle through Q, with the line OQ' as axis of symmetry. Then $\mathscr{K}' \subset \mathscr{S}$, and $\mathscr{K}'$ is the base circle of some cylinder $\mathscr{C}'$ which sits on $\mathscr{B}$ (by above) and $\mathscr{C}'$ does not sit flat on $\mathscr{B}$ since $\overset{0}{\mathscr{C}'}$ contains the point Q' in $\mathscr{S} \cap \mathrm{Fr}\,\mathscr{B}$. Hence, applying the above technique to Q' and $\mathscr{K}'$, the spherical cap of $\mathscr{S}$ with axis of symmetry OQ' which subtends a semi-vertical angle 2ψ at O is contained in $\mathrm{Fr}\,\mathscr{B}$. But as Q' was just an arbitrary point of $\mathscr{K}$, this shows that the spherical cap of $\mathscr{S}$ with axis of symmetry OQ and semi-angle 3ψ is contained in $\mathrm{Fr}\,\mathscr{B}$. But then this is true for Q' in place of Q. Hence the spherical cap of $\mathscr{S}$ with axis OQ and semi-angle 4ψ is contained in $\mathrm{Fr}\,\mathscr{B}$. Iteration of this technique shows at once that the whole of $\mathscr{S}$ is in $\mathrm{Fr}\,\mathscr{B}$. By convexity, the ball bounded by $\mathscr{S}$ is in $\mathscr{B}$. If $P \in \mathscr{B}$ and P lies outside $\mathscr{S}$, then the fact that all points on the lines from P to $\mathscr{S}$ belong to $\mathscr{B}$ contradicts the fact that $\mathscr{S} \subset \mathrm{Fr}\,\mathscr{B}$. Hence $\mathscr{B}$ is simply the ball bounded by $\mathscr{S}$.

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