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MINIMIZATION OF ± 1 *MATRICES UNDER LINE SHIFTS*

BY

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Consider the following device, which is due to E. R. Berlekamp of Bell Labs and Professor David Gale (University of California, Berkeley): an $n \times n$ array of lights is controlled by $2n$ switches, one for each row and one for each column. When a switch is thrown, all lights in the corresponding column which are off turn on, and those which are on turn off. This device may be thought of as a code with $2n-1$ information bits and minimum Hamming distance n. It is easy to enumerate the codewords by weight (i.e., the number of ways of getting K lights on starting from an all-off position), but it is not so easy to find the distribution of co-set leaders (i.e., the minimum number of lights left on starting from an arbitrary position). Finding co-set leaders for $n = 15$ was the object of a computer experiment of Gleason [1]. Moon and Moser [3] showed that, as n becomes large, most co-set leaders have weights close to $n^2/2$. The purpose of this note is to derive upper bounds on the weights of co-set leaders.

It is convenient to consider the array of lights as a matrix of ± 1s ($+1$: light on; -1: light off). Then throwing a switch simply corresponds to multiplying the corresponding row or column by -1. In this guise the problem we are addressing was posed by Tusnády and Van Lint. There is no inherent reason to restrict our attention to square matrices.

Let $C(m, n)$ denote the set of all $m \times n$ matrices with entries ± 1. Let $A = [a_{ij}] \in C(m, n)$. We are allowed to choose a subset of rows and change the signs of all the entries in these rows. We are also allowed to do the same with the columns. More formally we define

$$A \cong B \quad \text{iff } B = D_m A D_n,$$

where D_m, D_n are square diagonal matrices of orders m, n and entries ± 1 on the diagonal. Clearly $\cong$ is an equivalence relation. Let $\{A\}$ denote the equivalence class containing A. Set

$$d(A) = \sum_{i,j} a_{ij},$$

$$l(A) = \tfrac{1}{2}[d(A) + mn] = \{\#\text{ of } a_{ij} = +1\}.$$

We wish to find

$$g(m, n) = \max_{A \in C(m,n)} \min_{B \in \{A\}} l(B),$$

$$f(n) = g(n, n)$$

Komlós and Sulyok [2] found (for m, n sufficiently large) the value of

$$\max_{A \in C(m,n)} \min_{B \in \{A\}} |d(A)|.$$

They observed that the moment method (which we apply below) shows that for some c_1, c_2

$$\frac{n^2}{2} - c_1 n^{3/2} \leqslant f(n) \leqslant \frac{n^2}{2} - c_2 n^{3/2}.$$

Our main result is the following:

THEOREM.

$$\frac{n^2}{2} - \frac{n^{3/2}}{2} + o(n^{3/2}) \leqslant f(n) \leqslant \frac{n^2}{2} - \frac{n^{3/2}}{\sqrt{2\pi}} + o(n^{3/2}).$$

Part I. $f(n) \leqslant \dfrac{n^2}{2} - \dfrac{n^{3/2}}{\sqrt{2\pi}} + o(n^{3/2})$.

Proof. Define $r_i(A) = \sum_j a_{ij}$, $s_i(A) = |r_i(A)|$, and $l_i(A) = \frac{1}{2}[r_i(A) + n] = [\#$ of $a_{ij} = +1$ in the ith row$]$. Fix $A \in C(n, n)$. Let σ range over the 2^n possible column shifts. Then, regardless of A, the i-th row of A runs over all 2^n possible values. Thus

$$E_\sigma\big(s_i(A^\sigma)\big) = E(X),$$

where X is the distance from the origin after a random walk of n steps of ± 1. By a straightforward integration involving the normal curve we find that

$$E_\sigma\big(s_i(A^\sigma)\big) = \sqrt{n}\sqrt{\frac{2}{\pi}} + o(n^{1/2}).$$

Thus

$$E_\sigma\Big(\sum_{i=1}^{n} s_i(A^\sigma)\Big) = n^{3/2}\sqrt{\frac{2}{\pi}} + o(n^{3/2}).$$

So for some fixed σ

$$\sum_{i=1}^{n} s_i(A^\sigma) \geqslant n^{3/2}\sqrt{\frac{2}{\pi}} + o(n^{3/2}).$$

Upon A^σ we then apply operation τ, a shifting of each row to achieve the minimal number of $+$1s in that row. We have

$$\begin{aligned} l_i(A^{\sigma\tau}) &= \min[l_i(A^\sigma), n - l_i(A^\sigma)] \\ &= \tfrac{1}{2}[n - s_i(A^\sigma)] \end{aligned}$$

Thus

$$\begin{aligned} l(A^{\sigma\tau}) = \sum_{i=1}^{n} l_i(A^{\sigma\tau}) &= \sum_{i=1}^{n} \tfrac{1}{2}[n - s_i(A^\sigma)] \\ &\leqslant \frac{n^2}{2} - \frac{n^{3/2}}{\sqrt{2\pi}} + o(n^{3/2}). \end{aligned}$$

(Note: $1/\sqrt{2\pi} \sim .4$)

Part II. $\dfrac{n^2}{2} - \dfrac{n^{3/2}}{2} + o(n^{3/2}) \leqslant f(n)$.

Proof. A Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1 such that

$$HH^T = nI,$$

where I is the idetity matrix. Hadamard matrices have been studied extensively and we shall only use some of the elementary properties. We note that

$$(D_m H D_n)(D_m H D_n)^T = D_m H D_n D_n^T H^T D_m^T = nI,$$

so that $\{H\}$ consists solely of Hadamard matrices. First we show: If H is a Hadamard matrix of order n, $l(H) \geqslant n^2/2 - n^{3/2}/2$. For consider the rows of H as vectors $\vec{u}_1, \ldots, \vec{u}_n \in R^n$. They are mutually orthogonal of length $\sqrt{n}$. Set $\vec{v} = (1, 1, \ldots, 1, 1) \in R^n$. To find a lower bound $l(H)$ we first find a lower bound for

$$\sum_{i=1}^{n} \vec{v} \cdot \vec{u}_i.$$

Multiplying $\vec{v}$ and all the $\vec{u}_i$ by $n^{-1/2}$ affects this sum by a factor of n^{-1}. By a rotation of R^n we send

$$\begin{aligned} &\vec{u}_i \to \vec{\epsilon}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \text{ in } i\text{-th position}), \\ &\vec{v} \to \vec{v}^* = (v_1, v_2, \ldots, v_n) \quad |\vec{v}^*| = 1. \end{aligned}$$

By elementary calculus the minimum value of $\sum \vec{v}^* \cdot \vec{\varepsilon} = \sum v_i$ subject to the condition $|\vec{v}^*| = 1$ is achieved when $v_i = -n^{-1/2}$ for all i. Thus

$$\sum_{i=1}^{n} \vec{v}^* \cdot \vec{\varepsilon}_i \geqslant -\sqrt{n},$$

$$d(H) = \sum_{i=1}^{n} \vec{v} \cdot \vec{u}_i \geqslant -n^{3/2},$$

and so

$$l(H) \geqslant \frac{n^2}{2} - \frac{n^{3/2}}{2}.$$

This would complete the proof of Part II except for the fact that Hadamard matrices do not exist for all orders. It is known that Hadamard matrices of order $4^i 12^j$ exist. By the theory of simultaneous approximations, given $\delta > 0$ we can find an n_0 such that, if $n > n_0$, then there exists a pair of integers (i, j) such that $0 < n - 4^i 12^j < \delta n$. Thus we can find matrices of any sufficiently high order which are "almost" Hadamard. But in order to complete our proof we must show that the non-Hadamard portions of these matrices make a substantial contribution to the number of $+1$s under any combination of row and column shifts. To do this we employ a method used by Moon and Moser [3].

Consider an $s \times t$ matrix A, where $s \leqslant t$. It is not difficult to show that there are 2^{s+t-1} matrices in the equivalence class of A. Thus there are $2^{st-(s+t-1)}$ equivalence classes altogether. Each contains a matrix B such that $l(B) \leqslant g(s, t)$. Thus

$$2^{st-(s+t-1)} \leqslant \sum_{i=0}^{g(s,t)} \binom{st}{i}.$$

An application of Stirling's formula shows that

$$g(s, t) \geqslant \frac{st}{2} - \sqrt{st(s+t)\frac{\log 2}{2}} + o(s^{1/2}t)$$

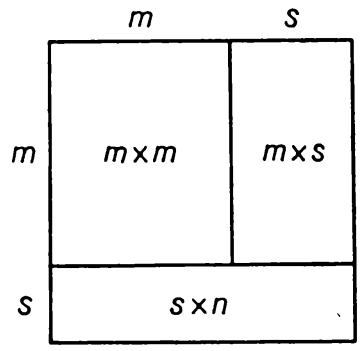
$$\geqslant \frac{st}{2} - ts^{1/2}\sqrt{\log 2} + o(s^{1/2}t).$$

Thus there exists a constant K such that

$$g(s, t) = g(t, s) \geqslant \frac{st}{2} - Ks^{1/2}t \quad \text{for all } t \geqslant s.$$

Given an $\varepsilon > 0$ pick n_0 such that for $n > n_0$ there exists an m such that an $m \times m$ Hadamard matrix exists and $m + s = n$, where $0 < s < \varepsilon^2 n/4K$.

Imagine constructing an $n \times n$ matrix from three "pieces", as indicated below:



Clearly

$$f(n) \geqslant f(m)+g(m, s)+g(s, n).$$

Applying our results above gives

$$f(n) \geqslant \frac{m^2}{2} - \frac{m^{3/2}}{2} + \frac{ms}{2} - Kms^{1/2} + \frac{ns}{2} - Kns^{1/2}$$

$$\geqslant \frac{(m+s)^2}{2} - \frac{m^{3/2}}{2} - K(m+n)s^{1/2}$$

$$\geqslant \frac{n^2}{2} - \frac{n^{3/2}}{2} - \varepsilon n^{3/2}.$$

This concludes the proof of part II.

Applying these methods we have found

$$f(1) = 0, \quad f(4) = 4, \quad 32 \leqslant f(10) \leqslant 37,$$
$$f(2) = 1, \quad f(5) = 7, \quad 72 \leqslant f(15) \leqslant 89.$$
$$f(3) = 2, \quad 22 \leqslant f(8) \leqslant 23,$$

We now study the values of $g(m, n)$ for n fixed and $m \to +\infty$.

THEOREM.

$$g(2^{n-1}, n) = \begin{cases} \sum_{i=0}^{k} i\binom{n}{i} & \text{if } n = 2k+1, \\ \sum_{i=0}^{k-1} i\binom{n}{i} + \frac{k}{2}\binom{n}{k} & \text{if } n = 2k. \end{cases}$$

Proof. We may think of an $A \in C(2^{n-1}, n)$ as corresponding to an ordered family of 2^{n-1} (possibly non-distinct) subsets of n-element set. In particular, $A \leftrightarrow (S_1, \ldots, S_{2n-1})$, where $S_j = \{i: a_{ij} = +1\}$. Define A by letting the 2^{n-1} sets run over all S, $|S| \leqslant k$ if $n = 2k+1$ and if $n = 2k$ all S, $|S| < k$ and all S, $|S| = k$, $1 \in S$.

A column shift preserves the family of S, except possibly if $n = 2k$ shifting from those S with $|S| = k$, $1 \in S$ to those with $1 \notin S$. Looking at Part I of our main theorem we note that $\sum_{i=1}^{n} s_i(A^\sigma)$ is independent of σ. As $E[\sum_{i=1}^{n} s_i(A^\sigma)]$ is independent of A this particular A gives the minimal possible $\max_\sigma \sum_{i=1}^{n} s_i(A^\sigma)$ and thus yields the minimal $g(n, 2^{n-1})$ which is as desired, q.e.d.

By the same argument

$$g(2^{n-1}k, n) = kg(2^{n-1}, n) \quad \text{for all integral } k \geqslant 1.$$

Setting

$$a_{n,p}(K) = g(2^{n-1}K+p, n) - Kg(2^{n-1}, n),$$

we can show

$$g(p, n) \leqslant a_{n,p}(K) \leqslant g(2^{n-1}, n)$$

and

$$a_{n,p}(K) \leqslant a_{n,p}(K+1),$$

As a is integral valued there exists $b_{n,p}$ such that

$$a_{n,p}(K) = b_{n,p} \quad \text{for all } K \geqslant K_{n,p}.$$

Thus, for fixed n, $g(m, n)$ is given for all sufficiently large m by 2^{n-1} linear equations of the form

$$g(2^{n-1}K+p, n) = Kg(2^{n-1}, n) + b_{n,p}.$$

For example

$$\begin{aligned}
g(2K, 2) &= K, \\
g(2K+1, 2) &= K, \\
g(4K, 3) &= 3K, \\
g(4K+1, 3) &= 3K, \\
g(4K+2, 3) &= 3K+1, \\
g(4K+3, 3) &= 3K+2, \\
g(8K, 4) &= 10K, \\
g(8K+1, 4) &= 10K, \\
g(8K+2, 4) &= 10K+2, \\
g(8K+3, 4) &= 10K+3, \\
g(8K+4, 4) &= 10K+4, \\
g(8K+5, 4) &= 10K+5, \\
g(8K+6, 4) &= 10K+7, \\
g(8K+7, 4) &= 10K+8.
\end{aligned}$$

These equations hold for all $K \geqslant 0$. It is possible that in general $g(2^{n-1}K+p, n) = Kg(2^{n-1}, n) + g(p, n)$ but we have not been able to prove this (**P 733**).

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