COLLOQUIUM MATHEMATICUM

VOL. XIII 1965 FASC. 2

THE CONSTRUCTION OF FINITE REGULAR HYPERBOLIC PLANES FROM INVERSIVE PLANES OF EVEN ORDER

BY

D. W. CROWE (MADISON, WISC.)

1. Introduction. The author has shown in [1] how to construct finite hyperbolic planes from the finite field $GF(q^2)$ of $q^2 = 2^{2t}$ elements. In fact, the planes constructed there are *finite regular planes of type* P_{k+m}^k $(k = 2^{t-1},\ m = 2^{t-1}+1)$ in the notation of Szamkołowicz [4]. A recent result of Dembowski [2] shows that the points and circles of an inversive plane of even order can always be represented as the points and (non-trivial) plane sections of an ovoid in the projective 3-space, $\mathscr{P}_3$, over $GF(2^t)$. This implies that any affine plane corresponding to such an inversive plane can be coordinatized by $GF(2^t)$, i. e., can be thought of as the "Argand diagram" for a $GF(2^{2t})$. Combined with the results of [1] this implies that a finite hyperbolic plane (and hence a P_{k+m}^k) can be constructed from any affine plane derived from the original inversive plane. The purpose of the present note is to show how to carry out this construction of a finite hyperbolic plane directly from an inversive plane of even order without recourse to the intermediate $\mathscr{P}_3$.

2. Definitions. An *inversive plane* $\mathscr{I}$ is a collection of *points* and distinguished subsets of points called *circles*, satisfying the axioms [2]:

(I.1) Three distinct points are contained in exactly one circle.

(I.2) If c is a circle containing point Q but not point P, then there is exactly one circle c' containing P and tangent to c at Q. (By definition, two point sets are *tangent* if and only if they have exactly one point in common.)

(I.3) There are at least two circles. Every circle has at least three points.

It can be verified in the standard way that if one circle contains exactly $n+1$ points, then every circle contains exactly $n+1$ points, every pair of points is contained in exactly $n+1$ circles, and there are exactly n^2+1 points in $\mathscr{I}$. The number n is called the *order* of $\mathscr{I}$.

The following Lemma 1 is an essential step in our construction.

LEMMA 1. *Let $\mathscr{I}$ be of even order n. If P is a point not on the circle c, then the circles containing P and tangent to c are exactly the $n+1$ circles containing P and some other point P^*.*

Proof. (In this proof we write "tangent circle" as an abbreviation for "tangent circle to c through P".) Let l be any circle through P and two points R, S of c. We first show that at each point ($\neq P$) of l there is exactly one tangent circle (to c, through P). At points R, S of l this is the content of (I.2). At any other point L ($\neq P$) of l we consider the $n-1$ circles determined by L, P and points ($\neq R, S$) of c. An odd number of these meet c in exactly one point, since $n-1$ is odd. That is, there is at least one tangent circle through L and P. Letting L range over l, this accounts for $n-2$ of the $n-1$ tangent circles (at points $\neq R, S$) to c. The remaining one cannot contain any L ($\neq P$), for then L would lie on an even number of such circles. Hence the remaining one contains only P.

This shows that no two tangent circles have another point of l in common. That is, through an intersection $P^* \neq P$ (if any) of two tangent circles there is no circle through P meeting c in two points. Hence, if some pair of tangent circles meets at $P^* \neq P$, then every circle through P, P^* and a point of c is a tangent circle. This accounts for all tangent circles, hence the required result. It only remains to prove that some pair of tangent circles actually intersects at P^*. This follows directly by counting the points on the $n+1$ tangent circles, assuming no two have points (except P) in common. This gives $n(n+1)+1$ points. But there are only n^2+1 points in the entire plane $\mathscr{I}$!

3. The hyperbolic plane. We can now construct a *finite hyperbolic plane* $\mathscr{H}$ from an inversive plane $\mathscr{I}$ of even order in the following way. Let Γ be a fixed circle of $\mathscr{I}$. By Lemma 1, each point P ($\notin \Gamma$) determines a unique point P^*, contained in each of the $n+1$ tangent circles to Γ through P. The *Points* of $\mathscr{H}$ are the pairs $\{P, P^*\}$. The *Lines* of $\mathscr{H}$ are the tangent circles to Γ. The Point $\{P, P^*\}$ is *incident* with the line l if and only if the point P is on the tangent circle l. We have only to verify that $\mathscr{H}$ satisfies the axioms for a finite hyperbolic plane [1]:

(H. 1) Two Points determine exactly one Line.

(H. 2) Through each Point not on Line l there are at least two Lines not meeting l.

(H. 3) If a subset, $\mathscr{S}$, of the Points of $\mathscr{H}$ contains three Points not on a Line, and contains all Points on Lines through pairs of Points of $\mathscr{S}$, then $\mathscr{S}$ contains all the Points of $\mathscr{H}$.

The verification of axioms (H. 1)-(H. 3) follows.

(H. 1) Let $\{P, P^*\}$ and $\{Q, Q^*\}$ be two points. The $n+1$ circles through P and Q include all points of $\mathscr{I}$, in particular P^*. The unique circle l through P, Q, and P^* is tangent to Γ, by Lemma 1. Hence l contains Q^*, and is the required Line.

(H. 2) At a Point $\{P, P^*\}$ there are exactly $n+1$ Lines. But each Line l contains exactly $n/2$ Points, since one point of the circle is on Γ, and the remaining n are identified in pairs. That is, there are exactly $n/2+1$ Lines at $\{P, P^*\}$ which fail to meet l.

In fact, this shows that $\mathscr{H}$ satisfies the stronger axiom

(H. 2′) Through each Point not on Line l there are exactly $m \geqslant 2$ Lines not meeting l.

If a finite plane satisfies (H. 1) and (H. 2′) (and contains at least 3 Points not on the same Line, and hence not all Points are on two Lines) then it is *regular*, i. e., all Lines contain the same number, k, of Points. For let l and l' contain k and k' Points respectively. Then at a Point $P \notin l \cup l'$ every Line either meets l or fails to meet l. That is, there are exactly $m+k$ Lines at P. Similarly there are $m+k'$ Lines at P. Hence $k = k'$.

Axiom (H. 3) is a consequence of

LEMMA 2. *A plane $\mathscr{H}$ satisfying* (H. 1) *and* (H. 2′) *necessarily satisfies* (H. 3) *if* $(k-1)^2 > m$.

Proof [3]. If $\mathscr{H}$ does not contain 3 Points not on a Line then (H. 3) is satisfied vacuously. So we can assume that $\mathscr{H}$ is regular, with k Points on each Line. Let $\bar{P}$, $\bar{Q}$, $\bar{R}$ be three points of $\mathscr{S}$. Then $\mathscr{S}$ contains the k Points of Line $\overline{PQ}$, and hence the $k(k-1)+1$ Points on Lines joining Points of $\overline{PQ}$ to $\bar{R}$. If $(k-1)^2 > m$ this is greater than $m+k$. But each Point of $\mathscr{H}$ is on exactly $m+k$ Lines. Thus each Point is on at least one Line containing at least two Points of $\mathscr{S}$. Hence every Point of $\mathscr{H}$ is in $\mathscr{S}$, as required.

In the present cases $k = n/2$ and $m = n/2+1$, so the condition $(k-1)^2 > m$ of Lemma 2, and hence axiom (H. 3), is satisfied when $n > 6$.

4. Inversive planes of odd order. If $\mathscr{I}$ is of odd order, coordinatized by a field, it can be shown (more or less as in [1]) that the circles *orthogonal* to a given circle yield a finite hyperbolic plane. (However, such a plane is not regular.) Alternatively, if $\mathscr{I}$ (of odd order) satisfies only (I. 1)-(I. 3), and has a sufficiently strong orthogonality relation for circles (in particular strong enough to guarantee the analog of Lemma 1), then a hyperbolic plane can be constructed from $\mathscr{I}$. This problem will be discussed elsewhere.

REFERENCES

[1] D. W. Crowe, *The trigonometry of $GF(2^{2n})$, and finite hyperbolic planes*, Mathematika 11 (1964), p. 83-88.

[2] P. Dembowski, *Inversive planes of even order*, Bulletin of the American Mathematical Society 69 (1963), p. 850-854.

[3] T. G. Ostrom, *Ovals and finite Bolyai-Lobachevsky planes*, American Mathematical Monthly 69 (1962), p. 899-901.

[4] L. Szamkołowicz, *On the problem of existence of finite regular planes*, Colloquium Mathematicum 9 (1962), p. 245-250.

Reçu par la Rédaction le 28. 10. 1964