COLLOQUIUM MATHEMATICUM

VOL. XIII 1965 FASC. 2

TWO IMPROVEMENTS OF A RESULT CONCERNING A PROBLEM OF K. ZARANKIEWICZ

BY

Š. ZNÁM (BRATISLAVA)

This article is a continuation of [5]. Here we want to correct two faults of [5]: owing to an incorrect quotation it is not clear that [5] is connected with [2]; moreover, in the first row of p. 82, $k = n^{-1}\sum_{i=1}^{n} k_i$ is to be replaced by $K = n^{-1}\sum_{i=1}^{n} k_i$.

Let A_n be a square matrix of order n, consisting exclusively of 1's and 0's. The problem of Zarankiewicz, which we consider here, is to determine the smallest number of 1's in A_n still ensuring the existence of a minor of order j, consisting exclusively of 1's, where j is a positive integer with

$$2 \leqslant j < n. \tag{1}$$

Let us denote the number sought for by $k_j(n)$. In [5] it is proved that

$$k_i(n) < 1 + [n(j-1)/2 + (j-1)^{1/j} n^{(2j-1)/j}], \tag{2}$$

where $[a]$ is the integral part of a.

This paper improves this result in two directions. We prove namely

ASSERTION I. *For $j \geqslant 2$ we have*

$$k_j(n) < 1 + [n(j-1)/2 + (j-1)^{1/j} n\bigl(n - \tfrac{3}{8}(j-1)\bigr)^{(j-1)/j}]. \tag{3}$$

ASSERTION II. *We have*

$$k_j(n) < 1 + [n(j-1)/\varepsilon + (j-1)^{1/j} n^{(2j-1)/j}], \tag{4}$$

where

$$\varepsilon = \frac{2\bigl(n/(j-1)\bigr)^{1/j} - 1}{\bigl(n/(j-1)\bigr)^{1/j} - 1}. \tag{5}$$

All cases we have examined show that Assertion II is stronger than Assertion I but we have not succeeded in proving this.

Estimation (3) is somewhat better than (2). Estimation (2) gives $k_3(10) < 69$ whereas (3) gives $k_3(10) < 66$.

From (5) it is obvious that $\varepsilon > 2$, hence (4) is better than (2). We have, for example, $k_3(8) < 48$ from (2) and $k_3(8) < 45$ from (4). Čulík [1] has shown that $k_3(8) = 43$. The numbers $k_3(n)$ for $n > 8$ are unknown. From (4) we can easily derive that $k_3(9) < 55$. Further from (4) we have $k_4(6) < 33$, $k_4(7) < 44$, $k_4(8) < 55$, $k_4(9) < 67$. It is very easy to show that $k_4(6) = 32$ and $k_4(7) = 43$, hence in these cases (4) offers the best estimation.

In order to derive (3) we need two lemmata.

LEMMA I. *If $2 \leqslant j < n$, then the inequality*

$$n^{2/j}\, U(U-j+1) > (j-1)^{2/j} n(n-j+1) \tag{6}$$

implies

$$n^{2/j}(U-r+1)(U-j+r) > (j-1)^{2/j}(n-r+1)(n-j+r) \tag{7}$$

for arbitrary U and r such that

$$1 \leqslant r < j. \tag{8}$$

Proof. After some modifications of (7) we get

$$\begin{aligned} n^{2/j}\, U(U-j+1) + n^{2/j}(r-1)(j-r) \\ > (j-1)^{2/j} n(n-j+1) + (j-1)^{2/j}(j-r)(r-1). \end{aligned} \tag{9}$$

Formulae (1) and (8) imply

$$n^{2/j}(r-1)(j-r) \geqslant (j-1)^{2/j}(r-1)(j-r). \tag{10}$$

Adding (6) and (10) we get (9), q. e. d.

LEMMA II. *For $j \geqslant 2$ we have $(5/8)^{j-1} > (1/2)^j$.*

The proof is evident.

In [5] it is proved that if

$$n\binom{U}{j} > (j-1)\binom{n}{j}, \tag{11}$$

then $k_j(n) < 1 + [nU]$.

For $j = 2$ formula (3) follows from [3]. To prove it for $j > 2$ it is sufficient to verify (11) for

$$U = (j-1)/2 + (j-1)^{1/j}\left(n - \tfrac{3}{8}(j-1)\right)^{(j-1)/j}.$$

We shall distinguish two cases.

1. If j is even, then (11) can be written in the form

$$\begin{aligned} &\left(n^{2/j}\, U(U-j+1)\right)\left(n^{2/j}(U-1)(U-j+2)\right)\ldots\left(n^{2/j}\left(U-(j-2)/2\right)(U-j/2)\right) \\ &> \left((j-1)^{2/j} n(n-j+1)\right)\left((j-1)^{2/j}(n-1)(n-j+2)\right)\ldots \\ &\qquad \ldots\left((j-1)^{2/j}\left(n-(j-2)/2\right)(n-j/2)\right). \end{aligned} \tag{12}$$

Owing to lemma II we get $U > j-1$, so all factors on both sides are positive. Hence, according to lemma I, in order to prove (12) it is sufficient to show the validity of (6).

Formula (1) implies that

$$n > (j-1)^{(j-2)/j}\big(n-\tfrac{3}{8}(j-1)\big)^{2/j}.$$

Multiplying this inequality by $\frac{1}{4}(j-1)^{(j+2)/j}$ we get after some modifications the inequality

$$\tfrac{1}{4}n(j-1)^{(j+1)/j}-\tfrac{1}{4}(j-1)^2\big(n-\tfrac{3}{8}(j-1)\big)^{2/j} > 0.$$

Now if we add $(j-1)^{2/j}n(n-j+1)$ to both sides of the last inequality, we get

$$(13)\qquad (j-1)^{2/j}n^2-\tfrac{3}{4}(j-1)^{(j+2)/j}n-\tfrac{1}{4}(j-1)^2\big(n-\tfrac{3}{8}(j-1)\big)^{2/j} > (j-1)^{2/j}n(n-j+1).$$

Since $U > j-1$, (13) implies

$$\begin{aligned} n^{2/j}U(U-j+1) &> \big(n-\tfrac{3}{8}(j-1)\big)^{2/j}U(U-j+1) \\ &= (j-1)^{2/j}\big(n-\tfrac{3}{8}(j-1)\big)^2-\tfrac{1}{4}(j-1)^2\big(n-\tfrac{3}{8}(j-1)\big)^{2/j} \\ &> (j-1)^{2/j}(n-j+1)n; \end{aligned}$$

thus (6) has been proved.

2. For an odd j we must still prove that (6) implies

$$(14)\qquad n^{1/j}\big(U-(j-1)/2\big) > (j-1)^{1/j}\big(n-(j-1)/2\big).$$

For $r = (j+1)/2$ lemma I yields

$$n^{2/j}\big(U-(j-1)/2\big)^2 > (j-1)^{2/j}\big(n-(j-1)/2\big)^2$$

as well as (14), since all its factors are positive, q. e. d.

Now we shall prove Assertion II.

Similarly as in the previous part we can show that it is sufficient to prove (6) for

$$U = (j-1)/\varepsilon+(j-1)^{1/j}n^{(j-1)/j}.$$

Formula (5) implies $(1-\varepsilon)/(2-\varepsilon) = \big(n/(j-1)\big)^{1/j}$, hence

$$(2-\varepsilon)n^{1/j} = (1-\varepsilon)(j-1)^{1/j}.$$

Multiplying this expression by $n(j-1)^{(j+2)/j}/\varepsilon$ we get

$$(15)\qquad \frac{2-\varepsilon}{\varepsilon}(j-1)^{(j+1)/j}n^{(j+1)/j} = \frac{1-\varepsilon}{\varepsilon}(j-1)^{()j+2)/j}n.$$

Since $\varepsilon > 2$, we obtain

$$0 < \frac{\varepsilon-1}{\varepsilon^2} < \frac{1}{\varepsilon}, \qquad 0 < (j-1)^2 n^{2/j} < (j-1)^{(j+2)/j} n$$

and further

$$\frac{1-\varepsilon}{\varepsilon^2}(j-1)^2 n^{2/j} > -\frac{1}{\varepsilon}(j-1)^{(j+2)/j} n. \tag{16}$$

Using (15) and (16) we get

$$\frac{1-\varepsilon}{\varepsilon^2}(j-1)^2 n^{2/j} + \frac{2-\varepsilon}{\varepsilon}(j-1)^{(j+1)/j} n^{(j+1)/j} > -n(j-1)^{(j+2)/j},$$

and, adding $(j-1)^{2/j} n^2$ to both sides of this inequality we get

$$n^{2/j}\big((j-1)/\varepsilon + (j-1)^{1/j} n^{(j-1)/j}\big)\big((1-\varepsilon)(j-1)/\varepsilon + (j-1)^{1/j} n^{(j-1)/j}\big)$$
$$> n^2 (j-1)^{2/j} - n(j-1)^{(j+2)/j},$$

hence $n^{2/j} U(U-j+1) > (j-1)^{2/j} n(n-j+1)$, and we may use Lemma I as in the proof of (3).

REFERENCES

[1] K. Čulík, *Poznámka k problému K. Zarankiewicze*, Práce Brnenské Základny ČSAV 26 (1955), p. 341-348.

[2] T. Kővári, Vera T. Sós and P. Turán, *On a problem of K. Zarankiewicz*, Colloquium Mathematicum 3 (1954), p. 50-57.

[3] I. Reiman, *Über ein Problem von K. Zarankiewicz*, Acta Mathematica Academiae Scientiarum Hungaricae 9 (1958), p. 269-273.

[4] K. Zarankiewicz, *P 101*, Colloquium Mathematicum 2 (1951), p. 301.

[5] Š. Znám, *On a combinatorical problem of K. Zarankiewicz*, ibidem 11 (1963), p. 81-84.

Reçu par la Rédaction le 5. 6. 1964