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0-*TIGHT SURFACES WITH BOUNDARY AND THE TOTAL CURVATURE OF CURVES IN SURFACES*

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The well-known Fenchel inequality says that the total curvature of a closed curve in E^3 is greater than or equal to 2π. For knotted curves this can be sharpened. We consider unknotted curves bounding a simply connected region in some compact surface M in E^3. Under the assumption that M is 0-tightly immersed into E^3 we get a sharper version of the Fenchel inequality. The same result is obtained without global assumptions on M but under specific assumptions on the simply connected region bounded by the curve.

We consider a smooth immersion $f\colon M \to E^3$ of a compact connected surface with or without boundary into Euclidean space. As usual we define f to be 0-*tight* if almost all height functions have exactly one minimum. For closed surfaces M, 0-tightness is equivalent to tightness (minimal total absolute curvature) but in case $\partial M \neq \emptyset$ it is quite different (see [1] and [12]).

On the other hand, an arbitrary smooth closed curve $c\colon S^1 \to E^3$ satisfies the Fenchel inequality (cf. [3])

$$\int_c |k|\,ds \geqslant 2\pi,$$

where $|k|$ denotes the usual curvature of c, and equality characterizes plane convex curves. Sharper versions of this inequality hold for c being knotted (see, e.g., [9]).

Now, let us assume that c is *unknotted*, i.e. c is the boundary of a simply connected region D lying in some surface $f\colon M \to E^3$ such that $c = f|\partial D$ with $D \subseteq M$. The aim of the following considerations is to get sharper results than the Fenchel inequality in this case under certain assumptions involving the curvature of D or that of M.

Of course, there are convex plane curves bounding a simply connected region with arbitrarily large positive or negative curvature parts. But if we assume that the Gaussian curvature K is nonpositive inside of D (for instance: if D is a minimal surface spanned by c), then the Gauss-Bonnet formula leads to

$$\int_c |k|\,ds \geqslant \int_c k_g\,ds = 2\pi - \int_D K\,do = 2\pi + \int_D |K|\,do,$$

where k_g denotes the geodesic curvature, and equality implies that c is asymptotic.

Similarly, we have the following

PROPOSITION. *Assume that $D \subseteq M \setminus \partial M$ is a closed disc and $f\colon M \to E^3$ is 0-tight. Then the closed curve $c = f\,|\,\partial D$ satisfies*

$$\int_c |k|\,ds \geqslant 2\pi + \int_{D\cap\{K<0\}} |K|\,do,$$

where equality holds if and only if $f\,|\,M\setminus \mathring{D}$ is also 0-tight.

An example due to Rodríguez has shown that there exist 0-tight surfaces with nonplanar boundary curve c (see [11]).

A proof of this proposition can easily be obtained if we use the well-known fact (see [2], [4]-[6], [8], [12]) that, on one hand, the Euler characteristic of M can be expressed by the average of the alternating sum of the number of critical points of the height functions and that, on the other hand, the total absolute curvature of f is the average of the sum of these numbers. Then the inequality stated in the proposition is nothing but the inequality that the total absolute curvature of f is greater than or equal to the sum of the Betti numbers of M (see [7] for an extension of the proposition to higher dimensions).

In the following theorem we make more detailed assumptions on the curvature inside of D but no global assumptions on some ambient surface M:

THEOREM. *Let $f\colon D \to E^3$ be an embedding of a closed disc and assume that the set $D_0 := \{x \in D \mid K(x) = 0\}$ consists of finitely many disjoint, simple, smooth asymptotic curves which have at most isolated points with vanishing curvature and where all the pieces γ of D_0 starting and ending in ∂D at different points satisfy*

$$\int_\gamma |k|\,ds < 2\pi.$$

Then for $c = f\,|\,\partial D$ the inequality

$$\int_c |k|\,ds \geqslant 2\pi + \int_{K<0} |K|\,do \tag{1}$$

holds. Furthermore, equality in (1) *implies then that c consists of pieces which are either planar or asymptotic.*

Remark. Note that all assumptions on f are generically satisfied except two: the assumption that D_0 consists of asymptotic curves (this is the most essential assumption) and the last condition on the total curvature of the pieces of D_0 starting and ending in ∂D at different points. The proof will show that all the pieces are plane curves and that this last assumption could be replaced by the following: all the pieces have zero winding number of the tangent. In a local version of the theorem for "small" discs D inside of a given surface this last condition would be automatically satisfied.

Proof of the Theorem. We first observe that under the given assumptions each piece in D_0 with nonvanishing curvature is planar by the Beltrami-Enneper formula $\tau^2 = -K$ for the torsion of asymptotic curves (cf. [13], p. 101). On the other hand, this plane is the tangent plane of each point in that piece, which follows easily from the vanishing of the normal curvature. By continuity this holds also for isolated points with $k = 0$. Thus each component of D_0 is a planar curve. Hence the geodesic curvature k_g of such a curve is just the usual oriented curvature k.

Obviously, we can neglect the pieces in D_0 having end points in D, and a priori each of the other components is either closed lying in D or nonclosed starting from ∂D and ending at ∂D. We will see that the first case cannot occur.

Assume that there is a closed plane asymptotic curve with $K = 0$ bounding a simply connected region. Because of the assumed finiteness there is at least one such curve γ with $K > 0$ or $K < 0$ everywhere in its interior B. Clearly, $K < 0$ cannot occur because a point with maximal distance from the plane spanned by γ would be a point with nonnegative curvature. On the other hand, $K > 0$ cannot occur because the Gauss-Bonnet formula and the Hopf "Umlaufsatz" would imply (note that γ is simply closed by assumption)

$$2\pi > 2\pi - \int_B K do = \int_\gamma k_g ds = \int_\gamma k ds = 2\pi,$$

a contradiction.

So each component of D_0 starts in ∂D and ends in ∂D at different points (here we use the assumption that the curves have no double points), and $D \setminus D_0$ consists of simply connected regions, say $D_1, \ldots, D_n$. We can assume that $K > 0$ for $i = 1, \ldots, r$ and $K < 0$ for $i = r+1, \ldots, n$. The situation in D looks like in Fig. 1.

Now choose a suitable orientation for D, and let us denote the pieces of $\partial D_i \setminus \partial D$ by c_{ij}, and the exterior angles (which all are nonnegative) at the start point of c_{ij} by α_{ij}, and at the end point by β_{ij}.

Now let us consider the case $K > 0$ in D_i (i.e. $1 \leqslant i \leqslant r$). For the moment replace all (plane) pieces $f(c_{ij})$ by the straight-line segment c_{ij}^* between the end points, and denote by α_{ij}^* and β_{ij}^* the corresponding exterior angles (here we use the assumption that f is an embedding). This looks like in Fig. 2.

The total absolute curvature of ∂D_i with the straight-line segments instead of the c_{ij} is

$$(2) \qquad \int_{\partial D_i \cap \partial D} |k|\, ds + \sum_j (\alpha_{ij}^* + \beta_{ij}^*) \geqslant 2\pi,$$

where equality implies that it is planar and convex (this is a generalized version of the corresponding theorem of Fenchel for differentiable curves)

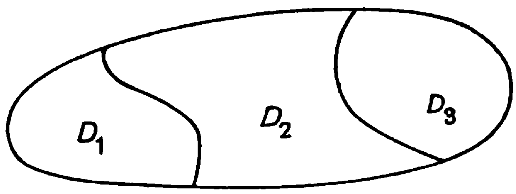

Fig. 1

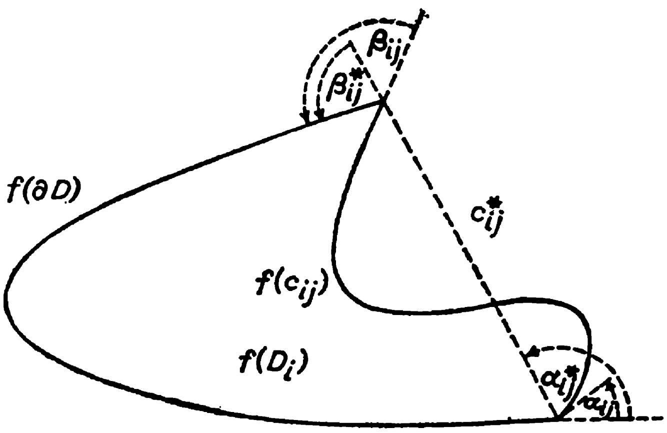

Fig. 2

On the other hand, the integral of curvature k over c_{ij} is just the difference of angles between the straight-line segment and c_{ij} (in this orientation) at the end point and the start point, i.e.

$$(3) \qquad \int_{c_{ij}} k\, ds = (\beta_{ij}^* - \beta_{ij}) - (\alpha_{ij} - \alpha_{ij}^*),$$

where we have used the assumption

$$\int_{c_{ij}} |k|\, ds < 2\pi$$

which implies that the winding number of the tangent of c_{ij} is zero. Thus from (2) and (3) we get the inequality

$$(4) \qquad \int_{\partial D_i \cap \partial D} |k|\, ds + \int_{\partial D_i \setminus \partial D} k\, ds \geqslant 2\pi - \sum_j (\alpha_{ij} + \beta_{ij}).$$

In case $K<0$ in D_i (i.e. $r+1 \leqslant i \leqslant n$) the Gauss-Bonnet theorem yields

$$(5) \qquad \int\limits_{\partial D_i \cap \partial D} |k|\,ds + \int\limits_{\partial D_i \setminus \partial D} k_g\,ds \geqslant \int\limits_{\partial D_i} k_g\,ds = 2\pi - \int\limits_{D_i} K\,do - \sum_j (\alpha_{ij}+\beta_{ij})$$

$$= 2\pi + \int\limits_{D_i} |K|\,do - \sum_j (\alpha_{ij}+\beta_{ij}).$$

Summing up (4) and (5) ($i = 1, \ldots, n$) by the equality $k_g = k$ (cf. above) each integral over a piece in $\partial D_i \setminus \partial D$ appears twice with different signs, so

$$\int\limits_{\partial D} |k|\,ds \geqslant n\cdot 2\pi + \int\limits_{K<0} |K|\,do - \sum_j (\alpha_{ij}+\beta_{ij})$$

$$= n\cdot 2\pi + \int\limits_{K<0} |K|\,do - 2(n-1)\pi = 2\pi + \int\limits_{K<0} |K|\,do,$$

which proves the asserted inequality.

Assume that equality holds. Then equality holds in (4) and (5) for each i. Hence the pieces of ∂D lying in $\{K>0\}$ are parts of planar convex curves, and the pieces of ∂D lying in $\{K=0\}$ are asymptotic by assumption, and hence planar (cf. above). Moreover, the pieces of ∂D lying in $\{K<0\}$ satisfy $|k| = k_g$, which implies that their normal curvature vanishes, and hence they are asymptotic.

Remark. If D lies in some closed orientable tight surface which in addition satisfies

$$(6) \qquad \operatorname{grad} K \neq 0 \quad \text{in } \{K=0\},$$

then by results of Nirenberg the assumptions of our theorem are satisfied (cf. [10]). In this case the asymptotic curves in $\{K<0\}$ touch tangentially the curves $\{K=0\}$.

Using this last property Rodríguez has shown (cf. [11], Theorem 21) that if M is a closed orientable tight surface satisfying (6) and if $N \subseteq M$ is a compact 0-tight surface with boundary, then ∂N consists of plane and convex curves. As a corollary we infer under assumptions that equality in (1) is impossible provided that $D \cap \{K<0\} \neq \emptyset$, i.e. in that case we have even the strict inequality

$$\int\limits_{c} |k|\,ds > 2\pi + \int\limits_{D\cap\{K<0\}} |K|\,do.$$

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